IMPACT LOADS ON BEAMS ON ELASTIC FOUNDATION

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By
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CERTIFICATE

Certified that this work "Impact Loads On Beams On Elastic Foundation", has been carried out by Sri Bharat Bhushan Prasad under my supervision and the same has not been submitted elsewhere for a degree.

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TO THE MEMORY
OF
MY MOTHER

ABSTRACT

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IMPACT LOADS ON BEAMS ON ELASTIC FOUNDATION

Modal Analysis and Energy Methods have been used to predict the response of beams on elastic foundation subjected to impact loads. Experiments have been conducted to verify these theoretical predictions, and they have been found in good agreement. The effect of foundation modulus and external damping on the beam response have been studied in detail. Displacement bounds have been computed within and outside elastic range for beams on elastic foundation for a certain class of dynamic loading and impact loading using Energy Theorems. Without going into the precise details of deformation process, the permanent deformations have been obtained for impact loads. The accuracy of these bounds have been checked with a few exact solutions. For a general class of dynamic loading as well as for impact loads, bounds on displacement and energy have been computed using variational principles. In a few cases, the values of displacement bounds have been compared

with the maximum displacement given by Modal Analysis and Energy Methods. The wave motion of a beam on elastic foundation in bending vibration after impact has been studied to predict how the flexural pulse and energy propagate in the beam. It has been observed that flexural pulse moves faster in a beam on elastic foundation than in a beam without foundation, where as energy moves at a slower rate along a beam on elastic foundation than along a beam without foundation.

CONTENTS

NOMENCLATURE

SYMBOLS	MEANING
$W(x,t)$	Beam Deflection
$\varepsilon(x,t)$	Bending Strain
$F(t)$	Impact Force
v_o	Striking Velocity of Striker
$v(x)$	velocity distribution along the beam
ρ	Mass Density of Beam Material
B	Width of Beam
H	Depth of Beam
m	Mass of the Beam per unit length
m_1	Total Mass of Beam
m_2	Mass of Striker
L	Length of Beam
A	Cross-section of Beam
ν	Poisson's Ratio
n	Show the States of Vibration of Beam
C	Damping Factor.
w_n	Eigen Frequency
X_n	Eigen Function
α_n	Eigen Value
K	Foundation Modulus
λL	Characteristic Length
E	Young Modulus of Elasticity

SYMBOLS	MEANING
I	Moment of Inertia of Beam
D	Dimensionless Displacement Parameter
M_o	Limiting Value of Bending Moment (Plastic Moment)
S	Dimensionless Impulse Parameter
σ_{ij}	Stress Tensor
$\dot{u}_i^o$	Initial Velocity
$\dot{u}_i t$	Velocity at any Time t
C^s	Complementary Strain Energy
T_i^s	Surface Traction
K^o	Initial Kinetic Energy
θ	Wave Length of Elastic Wave
v	Phase velocity
U	Group Velocity
K_g	Radius of Gyration
e	Co-efficient of Restitution
T_H	Hertz Time of Contact during Impact.
T_L	Contact Duration during Impact.
T	Kinetic Energy
V	Potential Energy
ξ	Damping Coefficient
t	Time.
V^*	Function containing two times the sum of Kinetic and Strain Energy.
F_i	Body Forces.

SYMBOLS	MEANING
$q(x,t)$	Dynamic loading
$\mathcal{L}$	Differential operator
$M(P)$	Mass Distribution function
$M(x,t)$	Bending Moment
$\sigma(x,t)$	Bending Stress
$S(x,t)$	Shear Force
$S(x)$	Bending Stiffness
$\mathcal{V}$	Denotes Volume
λ^*	Lagrange Undetermined Multiplier
X_o	Rigid Body Translational Mode
X_1	Rigid Body Rotational Mode
X_n^*	Elastic Mode
τ	Time Interval

LIST OF FIGURES

LIST OF TABLES

LIST OF PLATES

CHAPTER I

INTRODUCTION

1.1 TRANSVERSE IMPACT ON BEAMS :

During and after the second World war, scientists and engineers have shown an increasing interest in solution of problems concerned with the dynamic loading in general and Impact loading in particular. The dynamic loading effects on stationary structures eg. Railways, Bridges, Runways for landing of Aircraft, Naval structures etc., have been the subject of great interest. A lot of work has been done on the behaviour of beams, subjected to impact loads (Ref. 6,7,8,12,13,14,15,18,31,32,33), but for beams on elastic foundations, very little information is available in the existing literature.

The duration of impact, the impact force exerted, peak values of deflection, stresses, strains, bending moment and shear etc. are the quantities of interest for such a situation. In addition to other factors, the foundation modulus, the external damping etc. play an important role in this soil-structure interaction problem. In the immediate neighbourhood of impact point, the transient stress distribution is obviously three dimensional; the waves are not only propagated along the beam but are also reflected from all surfaces. At points farther away, the stresses will be more

uniformly distributed across the width, exhibiting the well known bending features with stress variation at a given cross-section occuring primarily as a function of depth. This pattern also extends to the region of contact point after removal of the impact load.

However, in order to make the problem amenable to theoretical treatment, it is assumed that the variation of stress across the width can be neglected. Thus finally the problem simplifies to a two dimensional problem of plane stress. Additional simplification is brought in by neglecting the rotatory inertia and transverse shear. The foundation model is that of Winkler [37], which consists of linear elastic springs.

1.2 LITERATURE REVIEW :

Lot of work has been reported on impact on beams, (Ref. 6,7,8,12,13,14,15,18,31,32,33), but very little information is available in literature for impact on beams on elastic foundation. Hoppmann [14,15] used energy method to predict the response of a simply supported beam with or without damping resting on Winkler model of foundation. He observed that with increase in the modulus of the elastic foundation supporting the beam reduces the amount of energy absorbed by the structure. He observed that this is due to the fact that the fundamental period of vibration of the structure becomes small in comparision with

time of impact. Impacts by relatively small masses moving with low velocities develope bending strain of practical concern.

For cases of impact on simply supported beams on elastic foundation, in which several modes are excited, the maximum deflection rapidly decreases with increase of foundation modulus whereas the strains decrease at a much smaller rate. If the fundamental mode of vibration of the beam is the only one appreciably excited by the impact the strain is proportional to the deflection and both decrease at the same rate with increasing stiffness of foundation [14].

External damping has a rapidly decreasing effect on reducing deflection and strain, as the number of the mode increases [14]. A very large portion of the deflection is due to the fundamental mode whereas a considerable portion of the strain is due to the higher modes. The co-efficient of restitution for the mass striking the beam increases very slowly with increase of foundation stiffness, if several modes of vibration are present.

1.3 SCOPE OF THE PRESENT INVESTIGATION :

Using Model Analysis and Energy Methods, the responses of Euler-Bernoulli beam resting on Winkler model of foundation to impact loads have been analysed. Chapter 2 deals with analytical investigations. The effect of external damping, foundation modulus on beam response have been studied. Experiments have been conducted to

verify the theoretical results. Chapter 5 deals with experimental set up and experimental measurements. The Euler-Bernoulli-beam on Winkler model has been simiulated in an experimental model consisting of an uniform mild steel beam and fairly linear uniform elastic springs.

For a certain class of dynamic loading including impact type of loading, displacement bounds for beams on elastic foundation have been computed using Energy Principles. The analysis of impact problems finally boils down to evaluation of initial velocity distribution or initial Kinetic Energy input. For a given initial velocity distribution, the displacement bounds within and outside elastic range have been computed in Chapter 3. These values have been compared with exact solutions for checking the accuracy of these bounds. The impact load is also characterized as a dynamic load $q(x,t)$ using Hertz Law of impact. The variational principles have been used in Chapter 3 to compute the displacement bounds of an Euler-Bernoulli beam resting on Winkler model of foundation, subjected to a general class of dynamic loading $q(x,t)$ including impact type of loading.

How the flexural waves propagate after the impact and how the energy propagates along a uniform beam on elastic foundation have been studied in Chapter 4. It has been shown that the energy propagates with a smaller velocity along beams on elastic foundation, as compared to beams without foundation. The general conclusions have been drawn in Chapter 6 along with scope for further research.

CHAPTER 2

ANALYTICAL INVESTIGATIONS

2.1. MODAL ANALYSIS (FREE FREE BEAM ON ELASTIC FOUNDATION WITHOUT DAMPING).

In this analysis beams on elastic foundation are considered without external damping. A method is developed wherein the vibration equation of beam on elastic foundation is solved and the applied impulse is replaced by a suitable initial condition. The free free beam resting on an elastic foundation is treated as undamped one dimensional continuous system. The vibration of such a system is represented by the partial differential equation

$$\text{£}\,[W(P,t)\,] + M(P)\ \frac{\partial^2 W(P,t)}{\partial t^2} = 0 \qquad (2.1.1)$$

where £ is a linear homogenous self-adjoint differential operator consisting of derivatives with respect to the spatial co-ordinate P, but not with respect to time, and M(P) is the mass distribution function. The above equation reduces to

$$\frac{\partial^2}{\partial x^2}\left[\ EI_{(x)}\ \frac{\partial^2 W(x,t)}{\partial x^2}\right] + K(x)\ W(x,t) + \rho A(x)\ \frac{\partial^2 W(x,t)}{\partial t^2} = 0$$

where the linear differential operator £ has the form

$$\text{£} = \frac{\partial^2}{\partial x^2}\,[EI_{(x)}\ \frac{\partial^2}{\partial x^2}\,] + K(x)$$

and $M(P) = \rho A(x)$.

For a uniform beam, on a uniform elastic foundation the above equation reduces to ,

$$EI \frac{\partial^4 W(x,t)}{\partial x^4} + \rho A \frac{\partial^2 W(x,t)}{\partial t^2} + KW(x,t) = 0 \qquad (2.1.2)$$

Here it can be shown that the operator $\pounds$ is only positive and not positive definite, whereas M is positive definite making it a semi-definite system. The solution of the above equation can be written as

$$W(x,t) = \sum_{n=0}^{\infty} X_n(x) \cdot q_n(t) \qquad (2.1.3)$$

And separating the variables in (2.1.2)

$$\frac{d^4 X_n}{dx^4} - \alpha_n^4 X_n = 0 \qquad (2.1.4)$$

and

$$\frac{d^2 q_n}{dt^2} + w_n^2 q_n = 0 \qquad (2.1.5)$$

where

$$_n^4 = (w_n^2/a^2) - (K/EI) \qquad (2.1.6)$$

$a^2 = EI/\rho . A$

X_n = Eigen function

w_n = Eigen frequency

α_n = Eigen value.

Solution of Equation (2.1.4) yields,

$$X_n = C \cdot e^{sx} \quad , \text{ where } s = \pm\ \alpha_n \ \text{ and } s = \pm\ i\,\alpha_n$$

satisfying the boundary conditions for free-free ends as

$$EI \left.\frac{\partial^2 W}{\partial x^2}\right|_{x=o} = EI \left.\frac{\partial^3 W}{\partial x^3}\right|_{x=o} = 0$$

$$EI \left.\frac{\partial^2 W}{\partial x^2}\right|_{x=L} = EI \left.\frac{\partial^3 W}{\partial x^3}\right|_{x=L} = 0 \qquad (2.1.7)$$

Then the frequency equation becomes

$$\text{Cos } \alpha_n L. \ \text{Cosh } \ \alpha_n L = 1 \qquad (2.1.8)$$

This transcendental equation has been solved numerically for the eigen values

n ---	α_{nL}
0	0
1	0
2	4.7300408
3	7.8532046
4	10.996078
5	14.1371655
6	17.278596

And for higher values of n, approximately for n > 6

$$\alpha_n = (2n-1)\frac{\pi}{2L} \qquad (2.1.9)$$

and $$w_n = \sqrt{\frac{EI}{\rho A}} \sqrt{\alpha_n^4 + (K/EI)} \tag{2.1.10}$$

Thus it can be observed that frequency equation (2.1.8) yields a zero double root.

$$\alpha_o = 0 \text{ and } \alpha_1 = 0$$

for $\alpha_n = 0$, equation (2.1.4) reduces to

$$\frac{d^4 X_n(x)}{dx^4} = 0 \tag{2.1.11}$$

which has the general solution

$$X_n = D_1 + D_2 x + D_3 x^2 + D_4 x^3 \tag{2.1.12}$$

It is easy to see that to satisfy the boundary conditions given by equation (2.1.7), $D_3 = D_4 = 0$. Hence corresponding to zero double roots, there are two modes

$$X_o(x) = A_o \tag{2.1.13}$$

$$X_1(x) = A_1(x/L - \tfrac{1}{2}) \tag{2.1.14}$$

where $X_o(x)$ is selected as symmetric and $X_1(x)$ as an anti-symmetric mode, as this particular linear combination renders the eigen function $X_o(x)$ and $X_1(x)$ orthogonal. They correspondingly represent rigid body transverse translation and rigid body rotation about the centre of mass of the beam. Since $\alpha_1 = \alpha_o = 0$ is a double root, any combination of $X_o(x)$ and $X_1(x)$ is also a natural mode. This explains why it is

possible to choose $X_1(x)$ to represent rotation about the centre of mass rather than any other point. Such unrestrained beams are usually treated as semi-definite system [23].

The frequency equation (2.1.8) further yields an infinite sequence of discrete eigenvalues, α_n ($n = 2, 3, \ldots \infty$) which correspond to the eigen-function.

$$X_n^*(x) = F_n [(\text{Cos } \alpha_n L - \text{Cosh } \alpha_n L)\ (\text{Sin } \alpha_n x + \text{Sinh } \alpha_n x) - (\text{Sin } \alpha_n L - \text{Sinh } \alpha_n L)(\text{Cos } \alpha_n x + \text{Cosh } \alpha_n x)] \tag{2.1.15}$$

It is interesting to note that eigen-values and eigen-functions are the same as for a free-free beam without elastic foundation. On the other hand the natural frequencies are given by equation (2.1.10) which are different from natural frequencies of the same beam without the elastic foundation. The effect of modulus of foundation diminishes as the mode number increases. Thus $X_n(x) = X_o + X_1 + X_n^*(x)$. Here $X_n(x)$ satisfies the boundary condition as well as the governing differential equation. As this semi-definite system has the mass distribution function $M(P) = \rho A(x) = m(x) = m$ which is positive definite, and as this is a self-adjoint problem, and $X_n(x)$ is also an admissible function , then [23]

$$\int_D m(P)\ X_n(P)\ X_k(P)\ dD(P) = \delta_{nk}$$

where δ is a Kronecker delta
D denotes the domain.

for $n = k, \quad \delta_{nk} = 1$

$n \neq k, \quad \delta_{nk} = 0$

and X_k is orthogonal to X_n

$$\int_0^L m \; X_n(x)\, X_k(x)\, dx = \delta_{nk}$$

For $X_n(x) = X_o = A_o$

$$\int_0^L m \, A_o \cdot A_o \cdot dx = 1$$

$$A_o = \pm \frac{1}{\sqrt{mL}} \qquad (2.1.16)$$

Again for $X_n(x) = X_1 = A_1(\frac{x}{L} - \frac{1}{2})$

then

$$\int_0^L m \; A_1^2 \left(\frac{x}{L} - \tfrac{1}{2}\right)^2 dx = 1$$

$$m . A_1^2 \; .(L/12) = 1$$

$$A_1 = \pm \sqrt{\frac{12}{mL}} \qquad (2.1.17)$$

Similarly for elastic modes, $X_n^* = F_n \cdot \bar{X}_n(x)$

$$\int_0^L m(X_n^*)^2 dx = 1$$

$$F_n = \pm \frac{1}{\sqrt{mL}} \quad \frac{1}{\overline{(\text{Sin}\ \alpha_n L - \text{Sinh}\ \alpha_n L)}}$$

$$X_n(x) = \frac{1}{\sqrt{mL}} + \sqrt{\frac{12}{mL}}\left(\frac{x}{L} - \frac{1}{2}\right) + \frac{1}{\sqrt{mL}}\ \{\ (\cos\alpha_n x + \cosh\alpha_n x) -$$

$$\left(\frac{\cos\alpha_n L - \cosh\alpha_n L}{\sin\alpha_n L - \sinh\alpha_n}\right)(\sin\alpha_n x + \mathrm{Sinh}\,\alpha_n x) \quad \ldots. \quad (2.1.18)$$

Eq.(2.1.5) yields the following solution,

$$q_n(t) = C_n \cos w_n t + D_n \,\mathrm{Sin}\, w_n\, t$$

So
$$W(x,t) = \sum_{n=2}^{\infty} X_n(x)\ \{C_n \cos w_n t + D_n \sin w_n t\ \} \qquad (2.1.19)$$

With initial condition

$$W(x,0) = 0$$

$$\left.\begin{aligned} \frac{\partial W}{\partial t} &= 0 \text{ for } t = 0,\ x \neq L/2 \\ \frac{\partial W}{\partial t} &= v_o \text{ for } t = 0,\ x = L/2 \end{aligned}\right| \qquad (2.1.20)$$

From Eq. (2.1.20) the constant $C_n = 0$

So Equation (2.1.19) reduces to;

$$W(x,t) = \sum_{n=2}^{\infty} D_n\, X_n(x) \sin w_n t \qquad (2.1.21)$$

Defining $\quad G_n = w_n \cdot D_n$

$$W(x,t) = \sum_{n=2}^{\infty} \frac{1}{w_n} \cdot G_n\, X_n(x)\ \mathrm{Sin}\, w_n t \qquad (2.1.22)$$

The evaluation of the constant G_n actually involves the replacement of the impulsive forces by an appropriate initial velocity condition for the beam [8]. If the striker is considered to impart a velocity indistinguishable from $\mathbf{v}_o$, to an infinitesimal section of the beam just under the contact point, then the momentum of this section and the striker should equal the initial momentum $m_2\mathbf{v}_o$, that is

$$\int \frac{\partial W}{\partial t} <x, 0> d\overline{Q} = m_2 \mathbf{v}_o \tag{2.1.23}$$

where integration $d\overline{Q}$ is carried out with respect to the total mass of the striker and beam

$$\frac{\partial W}{\partial t}(x,0) = \sum_{n=2}^{\infty} G_n X_n = \psi(x) \tag{2.1.24}$$

Multiplying both sides by another eigen function X_k

$$\sum_{n=2}^{\infty} G_n \cdot X_n \cdot X_k = \psi(x) \cdot X_k \tag{2.1.25}$$

If Eq.(2.1.25) is multiplied by m_2 and evaluated at $X = L/2$ and is added to the integral of Eq.(2.1.17) with respect to the beam mass $dm_1 = (m_1/L)dx$, then in view of symmetry, the resulting expression can be obtained as

$$\sum_{n=2}^{\infty} G_n \left[2\int_0^{L/2} \cdot X_n \cdot X_k (m_1/L)\, dx + m_2\, X_n(L/2)\, X_k(L/2) \right]$$
$$= \frac{2m_1}{L} \int_0^L \psi(x)\, X_k \cdot dx + m_2 \cdot \psi(L/2)\, X_k(L/2) \tag{2.1.26}$$

Eq. (2.1.19) represents really an integration with respect to the total mass as suggested by the Impulse-momentum equation (2.1.15). For the orthogonal modes, it can be shown that left hand side of Equation (2.1.26) vanishes for $n \neq k$, and hence

$$G_n = \frac{\frac{m_1}{L} \int_0^L \psi(x)\, X_n\, dx + m_2\, X_n(L/2)\, \psi(L/2)}{\frac{m_1}{L} \int_0^L X_n^2\, dx + m_2\, X_n(L/2)\, X_n(L/2)} \tag{2.1.27}$$

Using the initial condition given by Equation (2.1.14),

$$G_n = \frac{m_2 v_o\, X_n\, (L/2)}{\frac{m_1}{L} \int_0^L X_n^2\, dx + m_2\, X_n^2\, (L/2)} \tag{2.1.28}$$

and $\int_0^L X_n^2\, dx = L.\ \frac{\text{Cos}^2\, \alpha_n}{\text{Sin}^4\, \alpha_n L} (\text{Cosh}\, \alpha_n L - \text{Cos}\, \alpha_n L),\ m_1 = \rho AL$

$$W(x,t) = m_2 v_o \sum_{n=2}^{\infty} \frac{1}{w_n} \cdot \frac{X_n(x).\ X_n(L/2).\text{Sin}\, w_n t}{\rho AL \frac{\text{Cos}^2\, \alpha_n L(\text{Cosh}\, \alpha_n L - \text{Cosh}\, \alpha_n L)}{\text{Sin}^4\, \alpha_n L} + m_2 X_n^2\, (L/2)}$$

$$W(x,t) = m_2 v_o \sum_{n=2}^{\infty} \frac{1}{w_n} \cdot \frac{X_n(x) \cdot X_n\,(L/2)\;\; \sin w_n\, t}{\rho AL \dfrac{\cos^2 \alpha_n L}{\sin^4 \alpha_n L} (\cosh \alpha_n L - \cos \alpha_n L) + m_2 X_n^2\,(L/2)} \tag{2.1.29}$$

The impact force $F(t)$ is evaluated by knowing the acceleration from equation (2.1.21),

$$F(t) = -m_2 \cdot \frac{\partial^2 W}{\partial t^2} \cdot (L/2,\; t) \tag{2.1.30}$$

Bending Moment is given by

$$M(x,t) = -EI \;\frac{\partial^2 W(x,t)}{\partial x^2} = -EI(\alpha_n)^2\, W(x,t) \tag{2.1.31}$$

Shear force is given by

$$S(x,t) = -EI \;\frac{\partial^3 W(x,t)}{\partial x^3} = -EI\,(\alpha_n^3)\, W(x,t) \tag{2.1.32}$$

Bending strain is given by

$$\varepsilon\;(x,t) = \frac{H}{2} \;\frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.1.33}$$

where H = depth of beam.

The stress is given by

$$\sigma\,(x,t) = E.\; H/2. \;\; \frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.1.34}$$

2.2 MODAL ANALYSIS (FREE- FREE BEAM WITH DAMPING)

The equation of motion for the transverse vibration of beam on elastic foundation with external damping, can be expressed as

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\,\frac{\partial^2 W(x,t)}{\partial x^2}\right] + K(x)\,W(x,t) + \rho(x)\,A\,\frac{\partial^2 W(x,t)}{\partial t^2} + C\,\frac{\partial W(x,t)}{\partial t} = 0 \qquad (2.2.1)$$

where C = Damping factor.

For a uniform beam on a uniform homogeneous elastic foundation, Eq.(2.2.1) reduces to

$$EI\,\frac{\partial^4 W(x,t)}{\partial x^4} + C\,\frac{\partial W(x,t)}{\partial t} + \rho A\,\frac{\partial^2 W(x,t)}{\partial t^2} + KW(x,t) = 0 \qquad (2.2.2)$$

Separating the variables in Eq.(2.2.2)

$$\frac{d^4 X_n}{dx^4} - \alpha_n^4 X_n = 0 \qquad (2.2.3)$$

and

$$\frac{d^2 q_n}{dt^2} + \frac{C}{\rho A}\,\frac{dq_n}{dt} + w_n^2 q_n = 0 \qquad (2.2.4)$$

The solution of (2.2.3) will be same as Equations (2.1.7) and (2.1.8).

Putting $c = \frac{C}{\rho A}$, Equation (2.2.9) will yield two roots given by

$$q_{n_{1,2}} = \tfrac{1}{2}\{-c \pm \sqrt{c^2 - 4w_n^2}\} \qquad (2.2.5)$$

Further introducing

$$\xi = \frac{c}{2w_n} = \text{damping co-efficient}$$

then $$q_{1,2} = -\xi w_n \pm \sqrt{\xi^2-1}\, w_n \qquad (2.2.6)$$

Hence three conditions can arise

(a) $\xi > 1$, making the system over damped

(b) $\xi = 1$, the critically damped System.

(c) $\xi < 1$, underdamped system.

2.2(a) Overdamped System :

As ξ exceeds unity, the two roots remain on the real axis and are unequal. The equation for q_r becomes

$$q_n = A_n\, e^{(-\xi + \sqrt{\xi^2-1})w_n t} + B_n \cdot e^{(-\xi - \sqrt{\xi^2-1})w_n t}$$

or $$q_n = e^{-\xi w_n t}\,(C_n \operatorname{Cosh} \sqrt{\xi^2-1}\; w_n t + D_n \operatorname{Sinh} \sqrt{\xi^2-1}\; w_n t) \qquad (2.2.7a)$$

Equation (2.2.7a) represents aperiodic motion. So solution of (2.2.4) can be written as

$$W(x,t) \quad \sum_{n=2}^{\infty} X_n(x)\, e^{-\xi w_n t}\,(C_n \operatorname{Cosh} \sqrt{\xi^2-1}\; w_n t + D_n \operatorname{Sinh} \sqrt{\xi^2-1}\; w_n t) \qquad \ldots(2.2.8a)$$

Using the initial condition (2.1.20), evaluation of C_n and D_n is done exactly in the same way as done for the case without damping. Finally, the solutions can be written as

$$W(x,t) = m_2 v_o \sum_{n=2}^{\infty} \frac{1}{w_n} \frac{X_n(x).X_n(L/2) . e^{-\xi w_n t} . \text{Sinh} \sqrt{\xi^2-1}\, w_n t}{\rho AL \dfrac{\cos^2 \alpha_n L(\cosh \alpha_n L - \cos \alpha_n L)}{\sin^4 \alpha_n L} + m_2 X_n^2(L/2).} \tag{2.2.9a}$$

The impact force is evaluated by knowing the acceleration from equation (2.2.9a)

$$F(t) = -m_2 \frac{\partial^2 W}{\partial t^2}(L/2, t) \tag{2.2.10}$$

Bending Moment is given by

$$M(x,t) = - EI \frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.2.11}$$

Shear force is given by

$$S(x,t) = - EI \frac{\partial^3 W(x,t)}{\partial x^3} \tag{2.2.12a}$$

Bending strain is given by

$$\varepsilon(x,t) = \frac{H}{2} \frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.2.13a}$$

stress, $\sigma(x,t) = E . H/2 . \dfrac{\partial^2 W(x,t)}{\partial x^2}$ (2.2.14a)

2.2 (b) UNDERDAMPED SYSTEM :

As ξ is less than unity, the two roots are complex conjugates, the solution for q_n becomes,

$$q_n = (A_n e^{+iw_n \sqrt{1-\xi^2}\, t} + B_n e^{-i w_n \sqrt{1-\xi^2}\, t}) . e^{-\xi w_n t}$$

$$\text{or} \quad q_n = e^{-\xi w_n t}(C_n \operatorname{Cos} w_d t + D_n \operatorname{Sin} w_d t) \qquad (2.2.7b)$$

where $w_d = w_n \sqrt{1 - \xi^2}$.

So solution of Equation (2.2.4) can be written as

$$W(x,t) = \sum_{n=2}^{\infty} X_n(x)\, e^{-\xi w_n t} (C_n \operatorname{Cos} w_d t + D_n \operatorname{Sin} w_d t) \qquad (2.2.8b)$$

Using the initial condition given by Equations (2.1.13) and (2.1.14), evaluation of C_n and D_n is done exactly in the same way as done for overdamped system. So

$$W(x,t) = m_2 v_o \sum_{n=2}^{\infty} \frac{1}{w_d} \frac{X_n(x) . X_n(L/2)\, e^{-\xi w_n t} \operatorname{Sin} w_d t}{\rho AL \dfrac{\operatorname{Cos}^2 \alpha_n L (\operatorname{Cosh} \alpha_n L - \operatorname{Cos} \alpha_n L)}{\operatorname{Sin}^4 \alpha_n L} + m_2 X_n^2(L/2)}$$

(2.2.9b)

The impact force is evaluated by knowing acceleration from equation (2.2.9b)

$$F(t) = -m_2 \frac{\partial^2 W}{\partial t^2} (L/2,\ t)\ldots \tag{2.2.10b}$$

Bending moment is given by

$$M(x,t) = -EI \frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.2.11b}$$

Shear force is given by

$$S(x,t) = -EI \frac{\partial^3 W(x,t)}{\partial x^3} \tag{2.2.12b}$$

Bending strain is given by

$$\varepsilon\ (x,t) = \frac{H}{2} \frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.2.13b}$$

$$\sigma\ (x,t) = E.\ \frac{\partial^2 W(x,t)}{\partial x^2}\ .\ H/2 \tag{2.2.14b}$$

2.2.(c) CRITICALLY DAMPED SYSTEM

If the damping coefficient ξ equals unity, it corresponds to critical damping. The solution for q_n becomes

$$q_n = (A_n + B_n t)\ e^{-w_n t} \tag{2.2.7c}$$

The final solution can be written as

$$W(x,t) = m_2 \dot{v}_0 \sum_{n=2}^{\infty} \frac{1}{w_n} \frac{X_n . X_n\ (L/2)\ .t.\ e^{-w_n t}}{\rho AL\ \dfrac{\mathrm{Cos}^2 \alpha_n L(\mathrm{Cosh}\ \alpha_n L - \mathrm{Cos}\ \alpha_n L)}{\mathrm{Sin}^4 \alpha_n L} + m_2 X_n^{\ 2}(L/2)} \tag{2.2.9c}$$

The impact force is evaluated by knowing the acceleration from equation (2.2.9c)

$$F(t) = - m_2 \frac{\partial^2 W}{\partial t^2}(L/2,\ t) \tag{2.2.10e}$$

Bending Moment is given by

$$M(x,t) = - EI \frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.2.11c}$$

Shear force is given by

$$S(x,t) = -EI \frac{\partial^3 W(x,t)}{\partial x^3} \tag{2.2.12c}$$

Bending strain is given by

$$\varepsilon(x,t) = \frac{H}{2} \frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.2.13a}$$

$$\text{stress, } \sigma\,(x,t) = e.\ H/2.\ \frac{{}^2W(x,t)}{x^2} \tag{2.2.14a}$$

2.3 MODAL ANALYSIS (SIMPLY SUPPORTED BEAM) (without damping)

The equation of motion for the transverse vibration is same as given in Equations (2.1.1) and (2.1.2) with the boundary condition

$$\left.\begin{aligned} W &= 0, \quad x = 0 \\ W &= 0, \quad x = L \end{aligned}\right| \tag{2.3.1}$$

$$-EI \left.\frac{\partial^2 W}{\partial x^2}\right|_{x=0} = - EI \left.\frac{\partial^2 W}{\partial x^2}\right|_{x=L} = 0 \tag{2.3.2}$$

As obtained in the Section (2.2), the solutions are

$$X_n = \text{Sin}\,\frac{n\pi x}{L} \tag{2.3.3}$$

$$\alpha_n = \frac{n\pi}{L} \tag{2.3.4}$$

and

$$w_n = \left[\frac{EI}{\rho A}\right]^{1/2} \left[\frac{n^4\pi^4}{L^4} + \frac{K}{EI}\right]^{1/2} \tag{2.3.5}$$

$$q_n = C_n \,\text{Cos}\, w_n t + D_n \,\text{Sin}\, w_n t \tag{2.3.6}$$

So

$$W(x,t) = \sum_{n=1}^{\infty} \text{Sin}\,\frac{n\pi x}{L}\,(C_n \,\text{Cos}\, w_n t + D_n \,\text{Sin}\, w_n t) \tag{2.3.7}$$

With the help of initial condition, C_n, D_n are evaluated exactly in the same may as for Free-Free case. The final solution is

$$W(x,t) = m_2 v_o \sum_{n=1}^{\infty} \frac{1}{w_n} \,\text{Sin}\,\frac{n\pi x}{L}\, \frac{\text{Sin}\, n\pi/2}{\frac{\rho AL}{2} + m_2 \,\text{Sin}^2\, n\pi/2} \tag{2.3.8}$$

2.4 MODAL ANALYSIS (SIMPLY SUPPORTED WITH DAMPING)

(2.4a) For $\xi > 1$ making the system overdamped, referring to Eq.(2.2.4) the solution for q_n becomes

$$q_n = e^{-\xi w_n t}(C_n \,\text{Cosh}\, \sqrt{\xi^2-1}\, w_n t + D_n \,\text{Sinh}\, \sqrt{\xi^2-1}\, w_n t) \tag{2.4.1a}$$

With evaluation of constants, as for Free-Free case

$$W(x,t) = m_2 v_o \sum_{n=1}^{\infty} \frac{1}{w_n}\, \frac{\text{Sin}\,\frac{n\pi x}{L}\,\text{Sin}\,\frac{n\pi}{2}\, e^{-\xi w_n t} \cdot \text{Sinh}\, \sqrt{\xi^2-1}\, w_n t}{\frac{\rho AL}{2} + m_2 \,\text{Sin}^2\, n\pi/2}$$

(2.4.2a)

(2.4b) For $\xi < 1$, making the system underdamped, the solution for q_n becomes

$$q_n = e^{-\xi w_n t} (C_n \text{ Cos } w_d t + D_n \text{ Sin } w_d t) \qquad (2.4.1b)$$

where $w_d = w_n \sqrt{1-\xi^2}$

With evaluation of constants, as for free-free case

$$W(x,t) = m_2 v_o \sum_{n=1}^{\infty} \frac{1}{w_d} \frac{\text{Sin } \frac{n\pi x}{L} \text{ Sin } \frac{n\pi}{2} e^{-\xi w_n t} \text{ Sin } w_d t}{\frac{\rho AL}{2} + m_2 \text{ Sin}^2 n\pi/2}$$

(2.4.2b)

The impact force F(t) is evaluated by knowing the acceleration from Eq. (2.4.4)

Bending Moment is given by

$$-EI \frac{\partial^2 W(x,t)}{\partial x^2} = -EI(\alpha_n)^2 W(x,t) \qquad (2.4.3b)$$

Shear Force is given by

$$-EI \frac{\partial^3 W(x,t)}{\partial x^3} = - EI(\alpha_n)^3 W(x,t) \qquad (2.4.4b)$$

(2.4c) For $\xi = 1$, making the system critically damped, the solution for q_n becomes

$$q_n = (A_n + B_n\, t)\, e^{-w_n t} \qquad (2.4.1c)$$

With evaluation of constants, as for Free-Free case

$$W(x,t) = m_2\, v_o \sum_{n=1}^{\infty} \frac{1}{w_n} \frac{\sin \frac{n\pi x}{L} \sin \frac{n\pi}{2} .\, t.\, e^{-w_n t}}{\rho AL/2 + m_2 \sin^2 n\pi/2} \qquad (2.4.2c)$$

2.5 ENERGY METHOD (FREE FREE BEAM)

The impact is characterized here as time dependent load designated as $F(x,t)$ and the governing differential equation for a uniform Free-Free beam with external damping takes the form

$$EI \frac{\partial^4 W(x,t)}{\partial x^2} + \rho A \frac{\partial^2 W(x,t)}{\partial t^2} + C \frac{\partial W(x,t)}{\partial t} + KW(x,t) = F(x,t) \qquad (2.5.1)$$

And if impact happens to be in the middle of the beam, $F(x,t) = F(\frac{L}{2}, t)$, then the term $F(x,t)$ may be replaced by an equivalent point force which depends only on time.

As done earlier, the cross-sectional dimension of the beam is considered small in comparison with the length and the effect of shear deformation and rotatory inertia is neglected. The boundary conditions are

$$-EI \left.\frac{\partial^2 W}{\partial x^2}\right|_{x=0,L} = -EI \left.\frac{\partial^3 W}{\partial x^3}\right|_{x=0,L} = 0 \qquad (2.5.2)$$

The initial conditions are expressed as

$$\left.\begin{aligned} W(x,0) &= 0 \\ \left.\frac{\partial W}{\partial t}\right|_{t=0} &= 0 \end{aligned}\right| \qquad (2.5.3)$$

Neglecting contact deformation, vibration of the striker and the static deflection due to attachment of mass m_2, the solution of Eq. (2.5.1) is given by

$$W(x,t) = \sum_{n=0}^{\infty} X_n(x)\; q_n(t) \qquad (2.5.4)$$

where $X_n(x)$ are eigenfunctions satisfying Eq. (2.5.1) and Eq. (2.5.2), as obtained earlier in Eq. (2.1.18).

Here $q_n(t)$ is determined by substituting in the Lagrange's equation the value of Kinetic Energy T and potential energy V of the beam-foundation system. The Lagrange's equation is written as

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_n}\right] - \frac{\partial T}{\partial q_n} + \frac{\partial V}{\partial q_n} + \frac{\partial D}{\partial \dot{q}_n} = Q_n \qquad (2.5.5)$$

where D = Energy dissipated due to damping. (Dissipation function)

$$\dot{q}_n = \frac{\partial q_n}{\partial t} \quad \text{and} \quad \frac{d}{dx}(X_n) = X_n'(x)$$

Expressing

$$T = \frac{1}{2}\sum_{n=0}^{\infty} (\dot{q}_n)^2\; \rho A \int_0^L (X_n)^2\; dx \qquad (2.5.6)$$

$$V = V_{BEAM} + V_{FOUNDATION}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (q_n) \; EI \int_0^L (X_n'')^2 \, dx$$

$$+ \frac{1}{2} \, K \sum_{n=0}^{\infty} (q_n)^2 \int_0^L (X_n)^2 \, dx \qquad (2.5.7)$$

$$D = \frac{1}{2} \sum_{n=0}^{\infty} C(\dot{q}_n)^2 \int_0^L (X_n)^2 \, dx \qquad (2.5.8)$$

in the Lagrange's equation (2.5.5), the resulting equation is

$$\ddot{q}_n + \dot{q}_n \left[\frac{C}{\rho A}\right] + q_n \left[\frac{EI}{\rho A} \frac{\int_0^L (X_n'')^2 \, dx}{\int_0^L X_n^2 \, dx} + \frac{K}{\rho A}\right]$$

$$= \frac{F(t) \; X_n \; (L/2)}{\rho A \int_0^L X_n^2 \, dx} \qquad (2.5.9)$$

Using Laplace Transform and with initial condition given by Eq. (2.5.3)

$$W(x,t) = \frac{1}{\rho A} \sum_{n=0}^{\infty} \frac{X_n(L/2) \; X_n(x)}{\int_0^L X_n^2 \, dx} \int_0^t F(\tau) e^{-\delta_n (t-\tau)} \sin \theta_n (t-\tau) \, d\bar{\tau} \, . \qquad (2.5.10)$$

where $\int_0^L X_n^2 \, dx = L \dfrac{\cos^2 \alpha_n L}{\sin^4 \alpha_n L} (\cosh \alpha_n L - \cos \alpha_n L)$

$$\delta_n = \frac{1}{2} \frac{C}{\rho A} \qquad (2.5.11)$$

$$\theta_n = [w_n^2 - \delta_n^2]^{1/2} \qquad (2.5.12)$$

$\int_0^t F(\tau).\ e^{-\delta_n(t-\tau)}\ \text{Sin}\ \theta_n(t-\tau)\ d\bar{\tau}$ is Duhamel's (Convolution) integral whose evaluation depends upon $F(t)$. So it is necessary to develop an analytical expression for the contact force $F(t)$ in in terms of the known physical parameters, so that this Duhamel's integral may be evaluated. Based on Hertz [10], Law of impact, impact forces have been given by Lee [18], Zeener and Feshback [39]. Selecting

$$F(t) = m_2\ v_o(1+e)\ \frac{\pi}{2T_L}\ \text{Sin}\ \frac{\pi t}{T_L} \quad 0 < t < T_L$$

$$= 0 \qquad\qquad t > T_L \qquad (2.5.13)$$

where T_L = period of contact and e = co-efficient of restitution. $W(x,t)$ can be evaluated by putting $F(t)$ in Eq. (2.5.10) and performing the integration. But still the deflection $W(x,t)$ depends upon the unknown co-efficient of restitution 'e'. Formula should be known so that 'e' may be determined in terms of known physical parameters. This has been evaluated in the same way as suggested by Hoppmann [14] for simply supported beam. The contact period has been computed as suggeste. by Hertz and Raman [28]

$$T_L = 3.28\ \left[\frac{m_2^2}{v_o \cdot r}\ \frac{(1-\nu^2)^2}{E^2}\right]^{1/5} \qquad (2.5.14)$$

where r = radius of the striker. Finally,

$$W(x,t) = \frac{\pi\ m_2\ v_o(1+e)}{2\rho AL\ T_L}\ \sum_{n=0}^{\infty}\ \frac{X_n\ (L/2)}{\text{Cos}^2\ \alpha_n L}\ \frac{\text{Sin}^4\ \alpha_n L.\ I_n . X_n}{(\text{Cosh}\ \alpha_n L - \text{Cos}\ \alpha_n L)} \quad (2.5.15)$$

For $t < T_L$

$$I_n = \frac{e^{-\delta_n t}}{(w_n^2 - \frac{\pi^2}{T_L^2})^2 + \frac{4\pi^2 \delta_n^2}{T_L^2}} \{[\frac{2\delta_n \pi}{T_L} \text{Cos } \theta_n t + \frac{\pi}{T_L \theta_n}(2\delta_n^2 - w_n^2 + \frac{\pi^2}{T_L^2}) \text{Sin } \theta_n t]$$

$$- [\frac{2\delta_n \pi}{T_L} \text{Cos } \frac{\pi t}{T_L} + (w_n^2 - \frac{\pi^2}{T_L^2}) \text{ Sin } \frac{\pi t}{T_L}]\} \qquad (2.5.16)$$

For $t > T_L$

$$I_n = \frac{e^{-\delta_n t}}{(w_n^2 - \frac{\pi^2}{T_L^2} + \frac{4\pi^2 \delta_n^2}{T_L^2}} (P_n \text{ Cos } \theta_n t + J_n)$$

$$P_n = [2e^{\delta_n T_L} \frac{\delta_n \pi}{T_L} \text{Cos } \theta_n T_L - \frac{\pi e^{nT_L}}{T_L \theta_n} (2\delta_n^2 - w_n^2 + \pi^2/T_L^2) \text{ Sin } \theta_n T_L + \frac{2\pi \delta_n}{T_L}]$$

$$J_n = [\frac{\pi e^{\delta_n T_L}}{T_L \theta_n}(2\delta_n^2 - w_n^2 + \frac{\pi^2}{T_L^2}) \text{Cos } \theta_n T_L + 2e^{\delta_n T_L} \frac{\delta_n \pi}{T_L} \text{Sin } \theta_n T_L + \frac{\pi}{T_L \theta_n} \cdot R_n] \text{ Sin } \theta_n T_L$$

$$R_n = (2\delta_n^2 - w_n^2 + \pi^2/T_L^2) \qquad (2.5.17)$$

The Bending Moment is given by

$$M(x,t) = -EI \frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.5.18}$$

Shear Force is given by

$$S(x,t) = -EI \frac{\partial^3 W(x,t)}{\partial x^3} \tag{2.5.19}$$

Bending Strain is given by

$$\varepsilon(x,t) = \frac{H}{2} \frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.5.20}$$

The stress is given by

$$\sigma(x,t) = E. \frac{H}{2} . \frac{\partial^2 W(x,t)}{\partial x^2} \tag{2.5.21}$$

2.6 RESULTS AND DISCUSSION.

Numerical Results have been obtained for the cases discussed in sections (2.1 to 2.5) and are presented in Figs. (2.1 to 2.10) which show the variation of Deflection, Strain, Impact force, Stress with velocity of impact, v_o and with time. The summation of series have been done for first ten harmonics using IBM 7044. The convergence of series, Eq. (2.1.29),Eq. (2.2.9a), Eq. (2.2.9b), Eq. (2.2.9c) for free-free beam on elastic foundation was very good. However the convergence of series expressed by Eq. (2.3.8), Eq. (2.4.2a), Eq. (2.4.2b), Eq. (2.4.2c) for simply supported Beam on Elastic Foundation was not that good.

The effect of foundation modulus is to increase the natural frequencies of vibration of the beam, however this effect diminishes as the mode number increases. The natural frequencies of the free-free beam on elastic foundation has been listed in Table (5.1).

Increment in impact velocity, v_o, increases the various quantities, like, impact force, deflection, strain, stress etc. as shown in Fig. 2.1 to 2.10.

The effect of external damping has been shown in Fig. 2.1 to 2.10. In case of $\xi > 1$, the motion is no longer oscillatory, being exponentially decreasing with time, making the motion aperiodic. In case of $\xi < 1$, the motion is oscillatory with diminishing amplitude. The beam response is greatly reduced by the use of external damping.

The beam responses, (Deflection, Strain, Stress etc.) as predicted by Model Analysis are higher than that obtained by Energy method for Free-Free beam as well as simply supported beam as shown in Fig.(2.1 to 2.10). The beam response for simply supported beam by Modal Analysis has been compared with Hoppmann's results (Energy Method) in Fig. (2.9). Here also the results by Modal Analysis are higher than by the Energy Method.

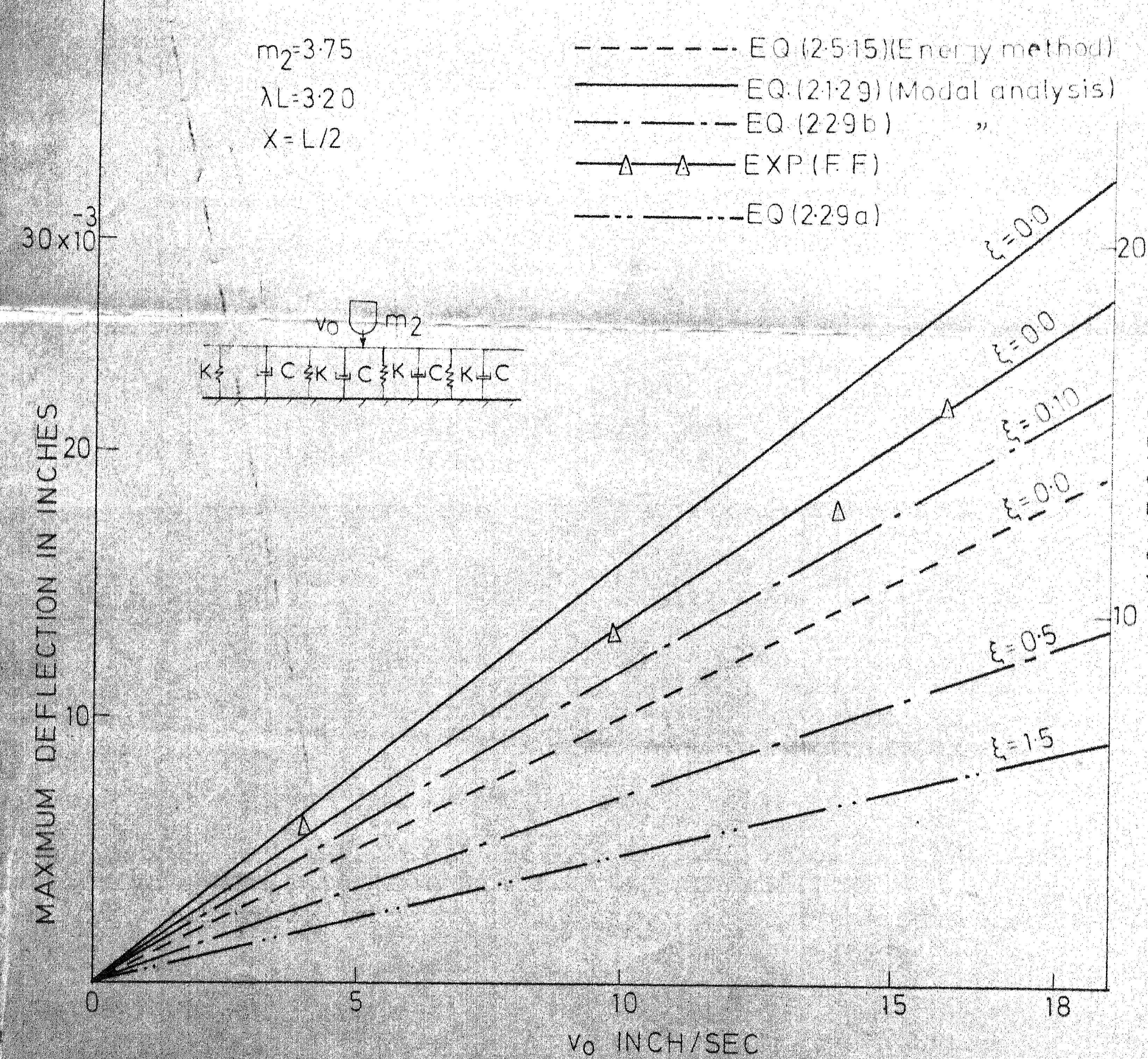


FIG 2·1 VARIATION OF MAXIMUM DEFLECTION WITH v_o

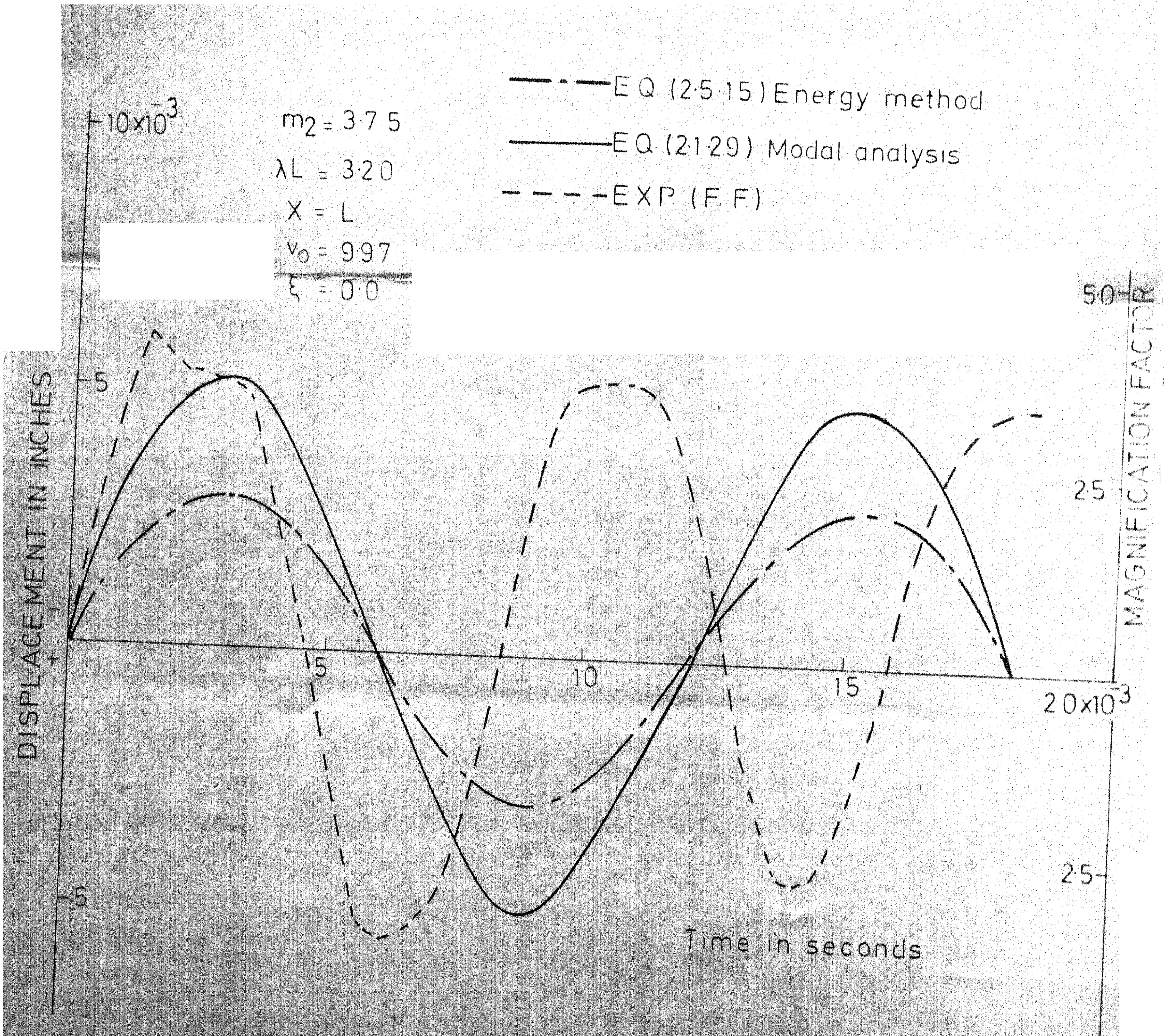


FIG. 2·1a VARIATION OF DEFLECTION WITH TIME

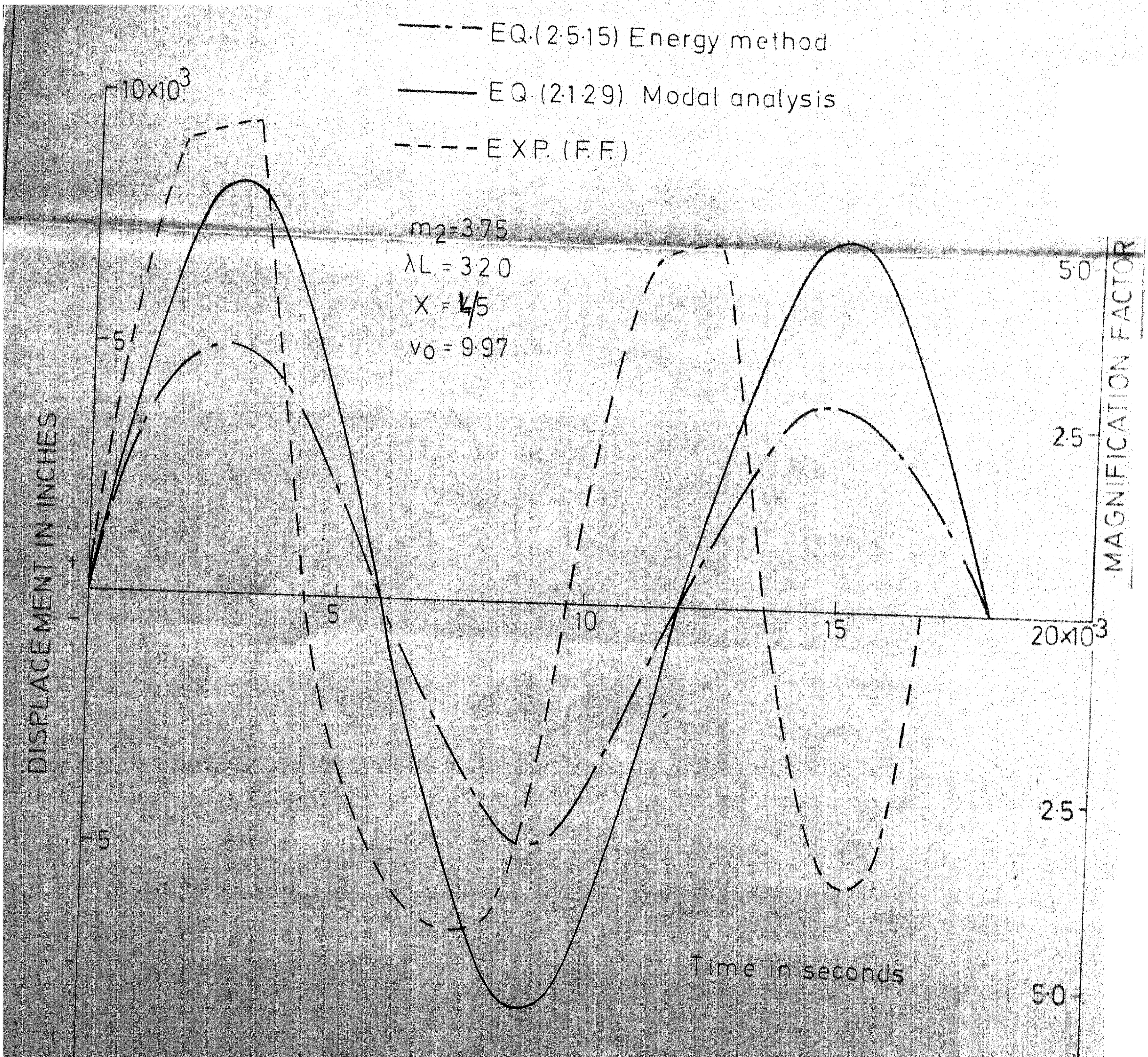


FIG. 2.1b VARIATION OF DEFLECTION WITH TIME

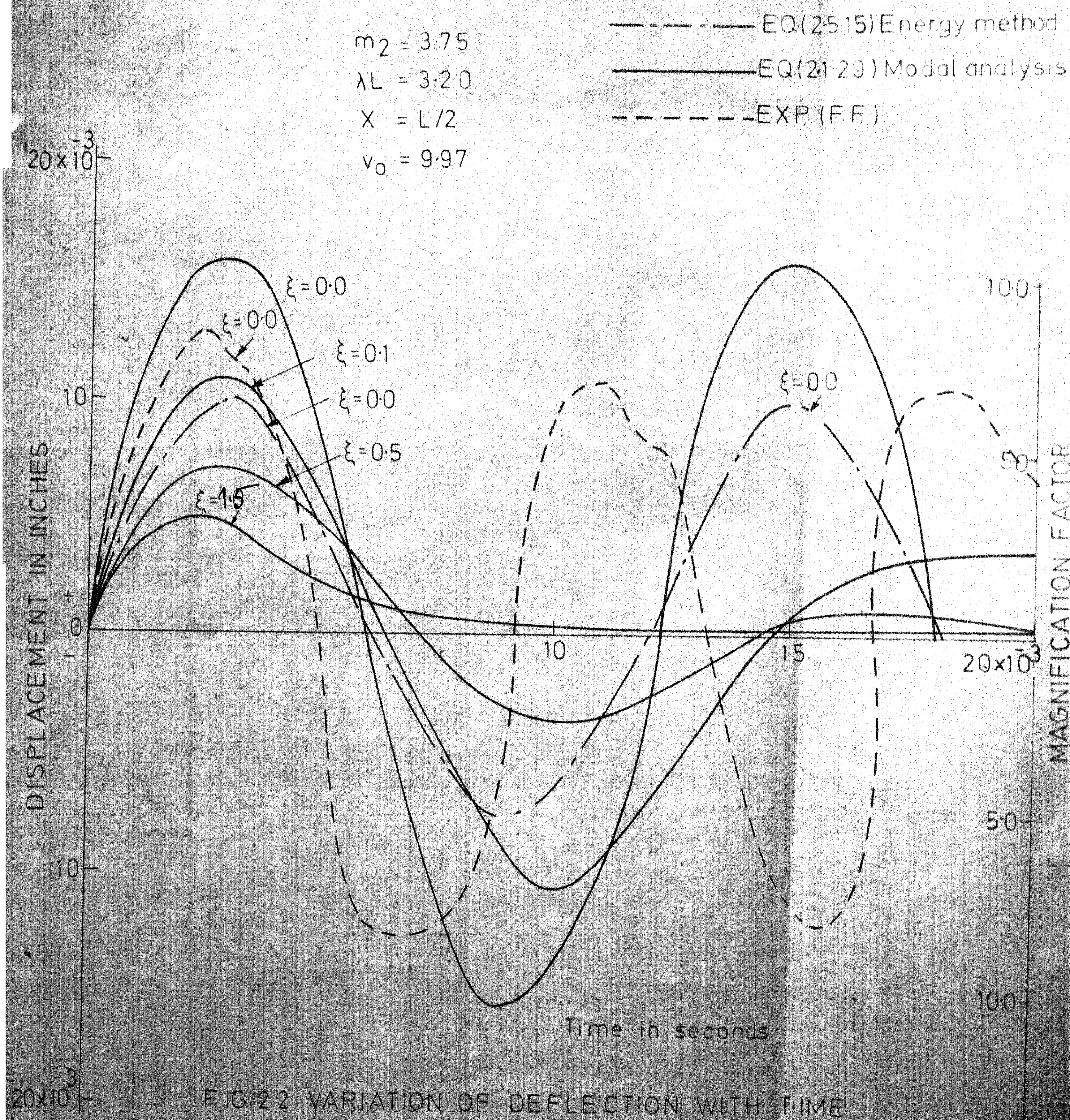


FIG.2.2 VARIATION OF DEFLECTION WITH TIME

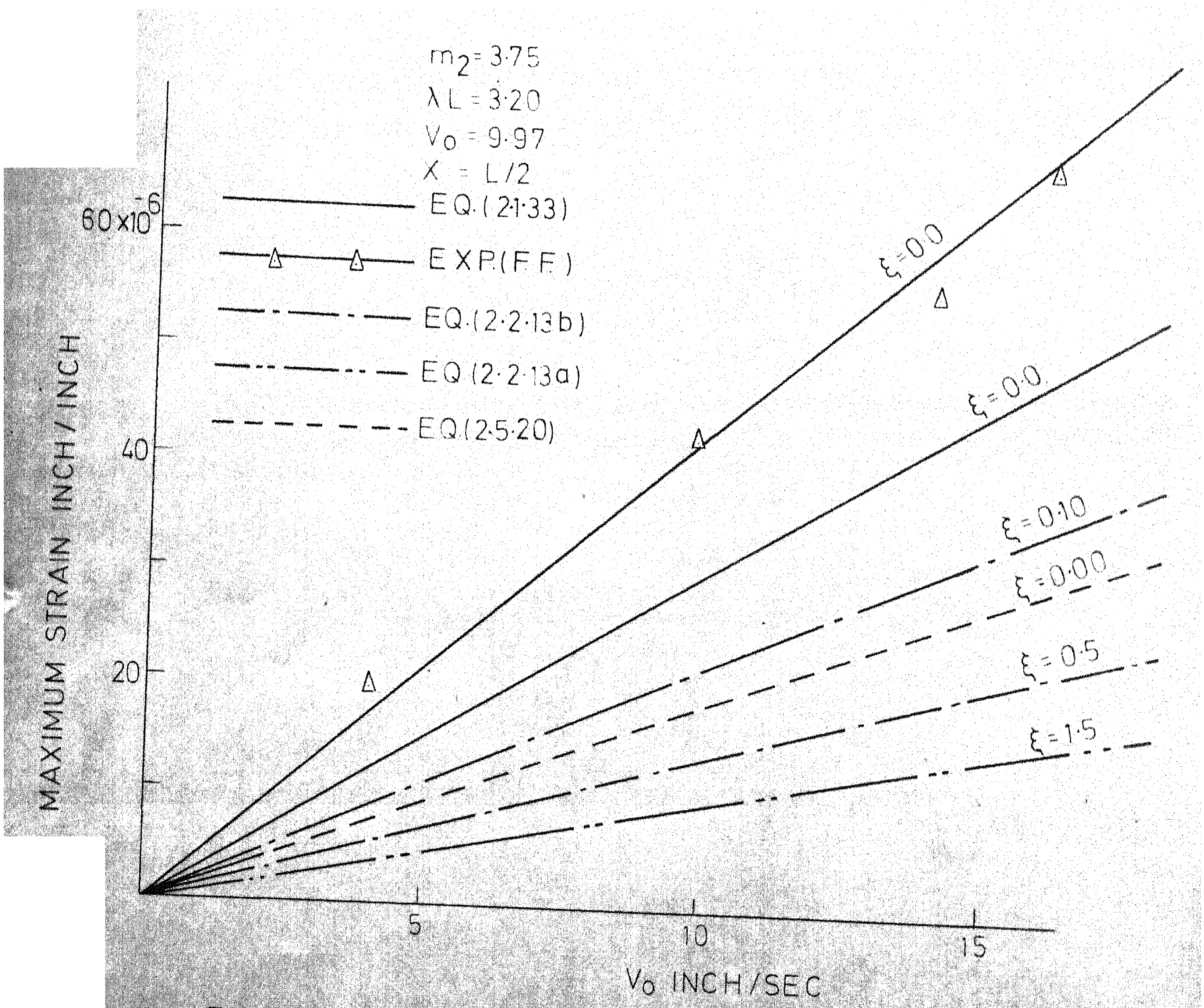


FIG. 23 VARIATION OF MAXIMUM STRAIN WITH v_o

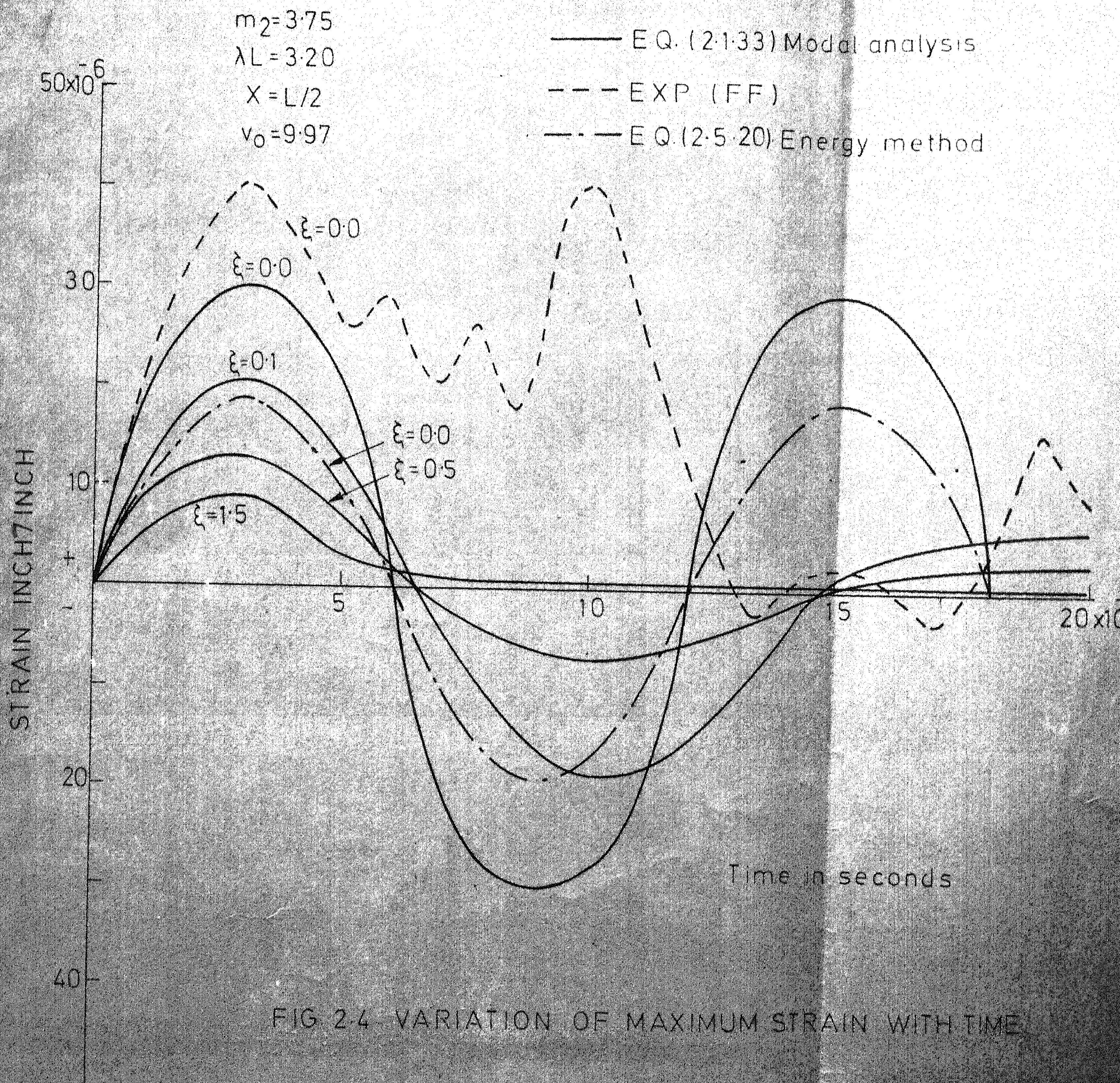


FIG 2·4 VARIATION OF MAXIMUM STRAIN WITH TIME

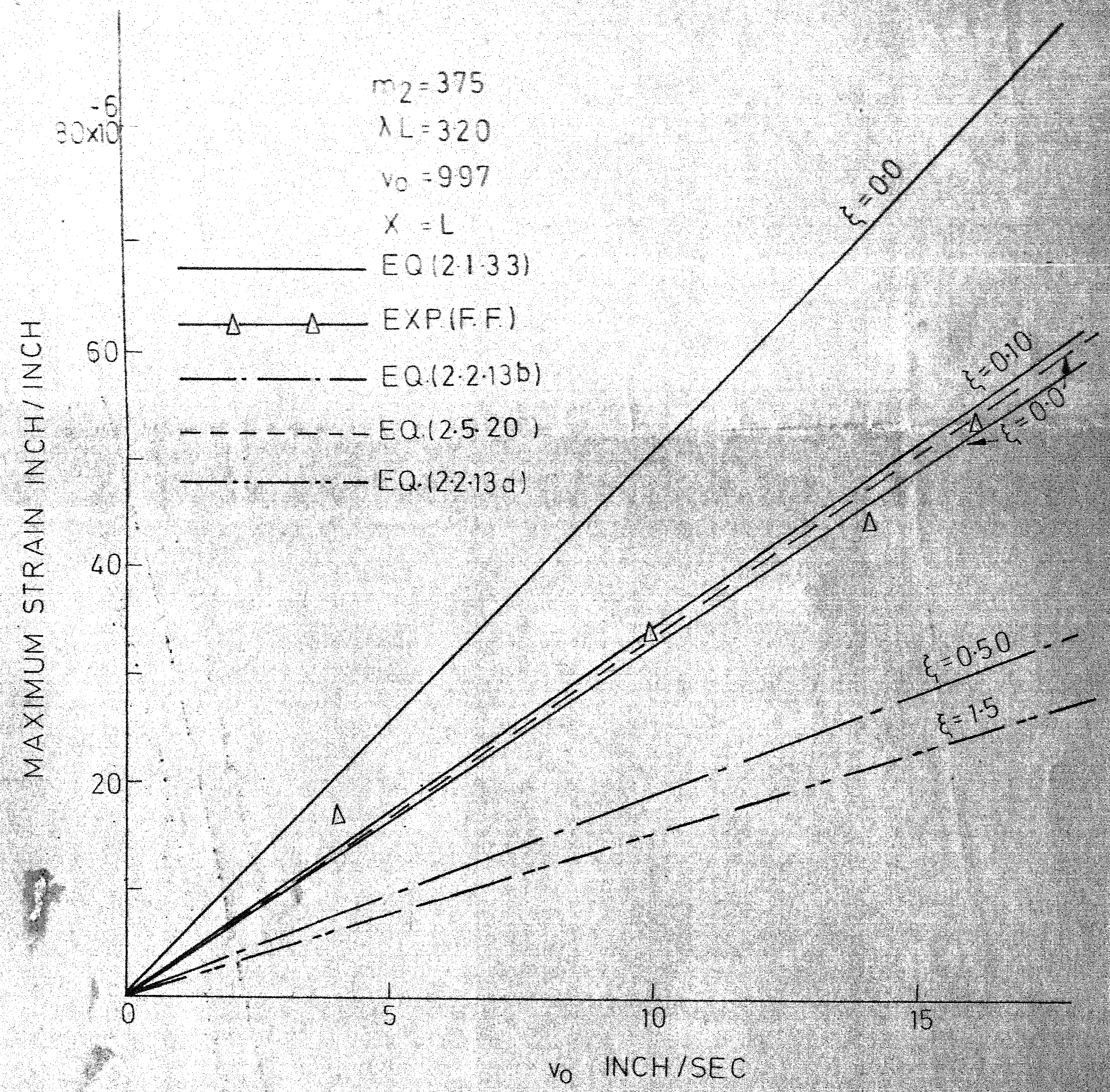


FIG. 2·5. VARIATION OF MAXIMUM STRAIN WITH v_o

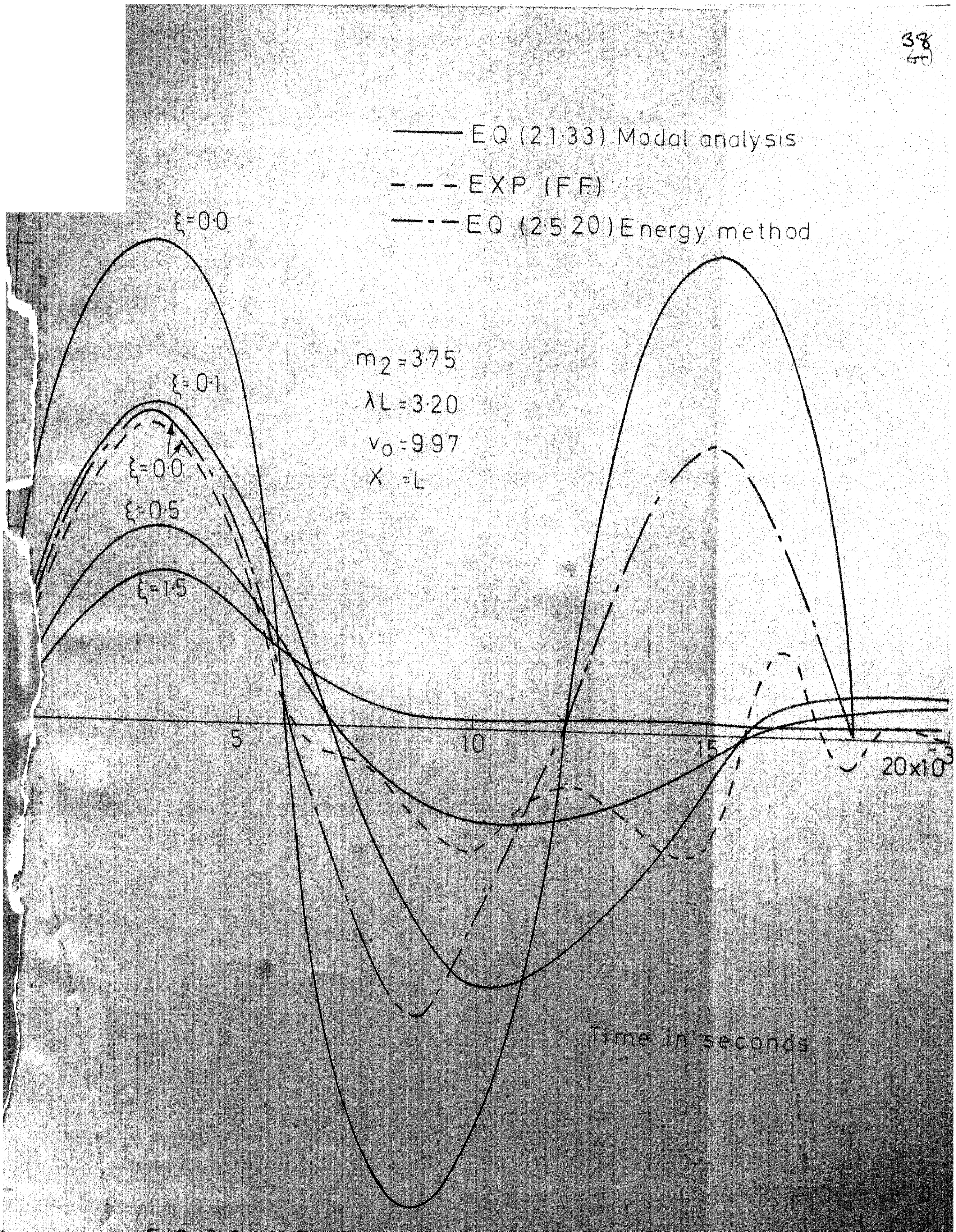


FIG. 2·6 VARIATION OF STRAIN WITH TIME

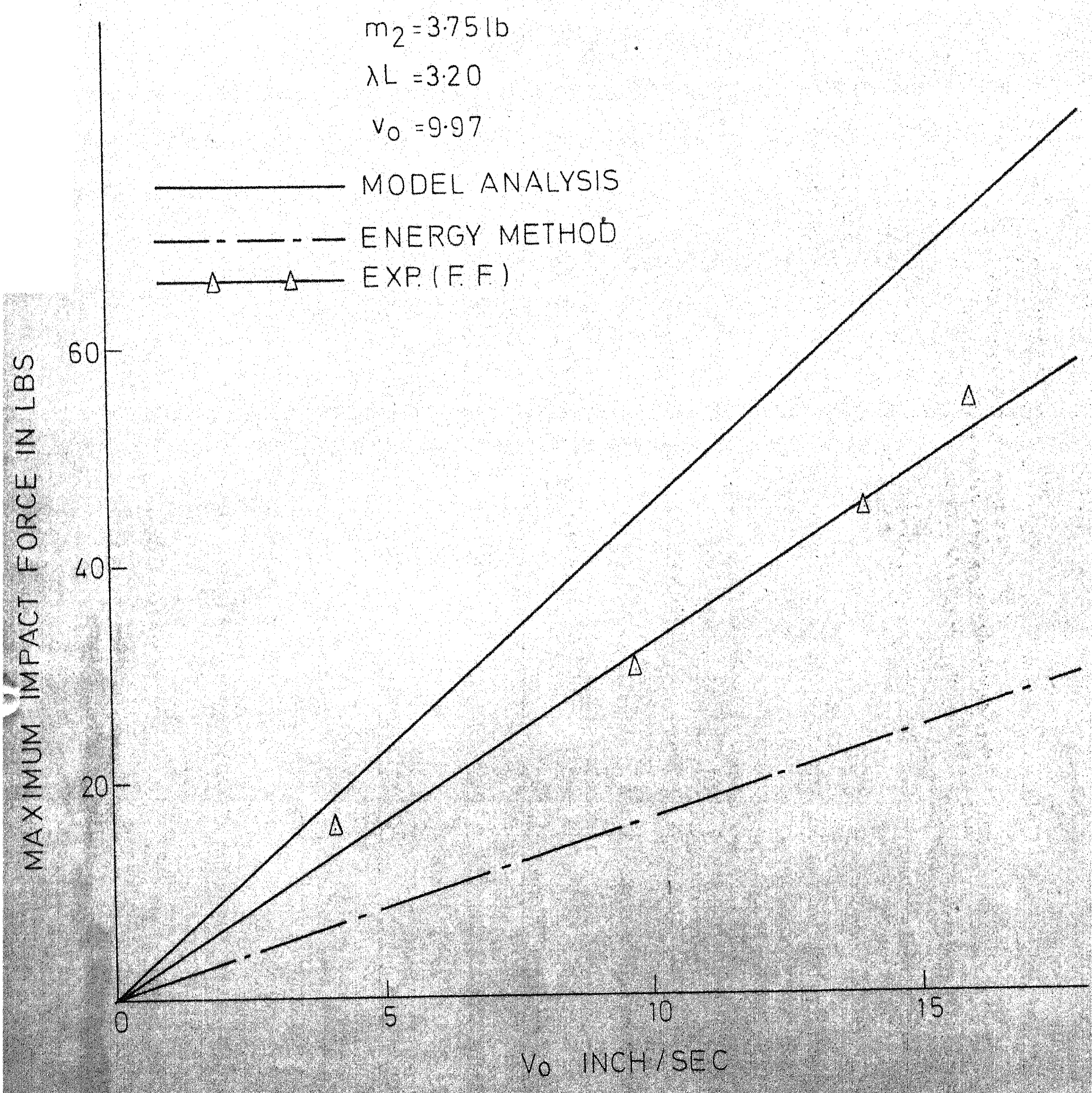


FIG. 2.7 VARIATION OF MAXIMUM IMPACT FORCE WITH V_o

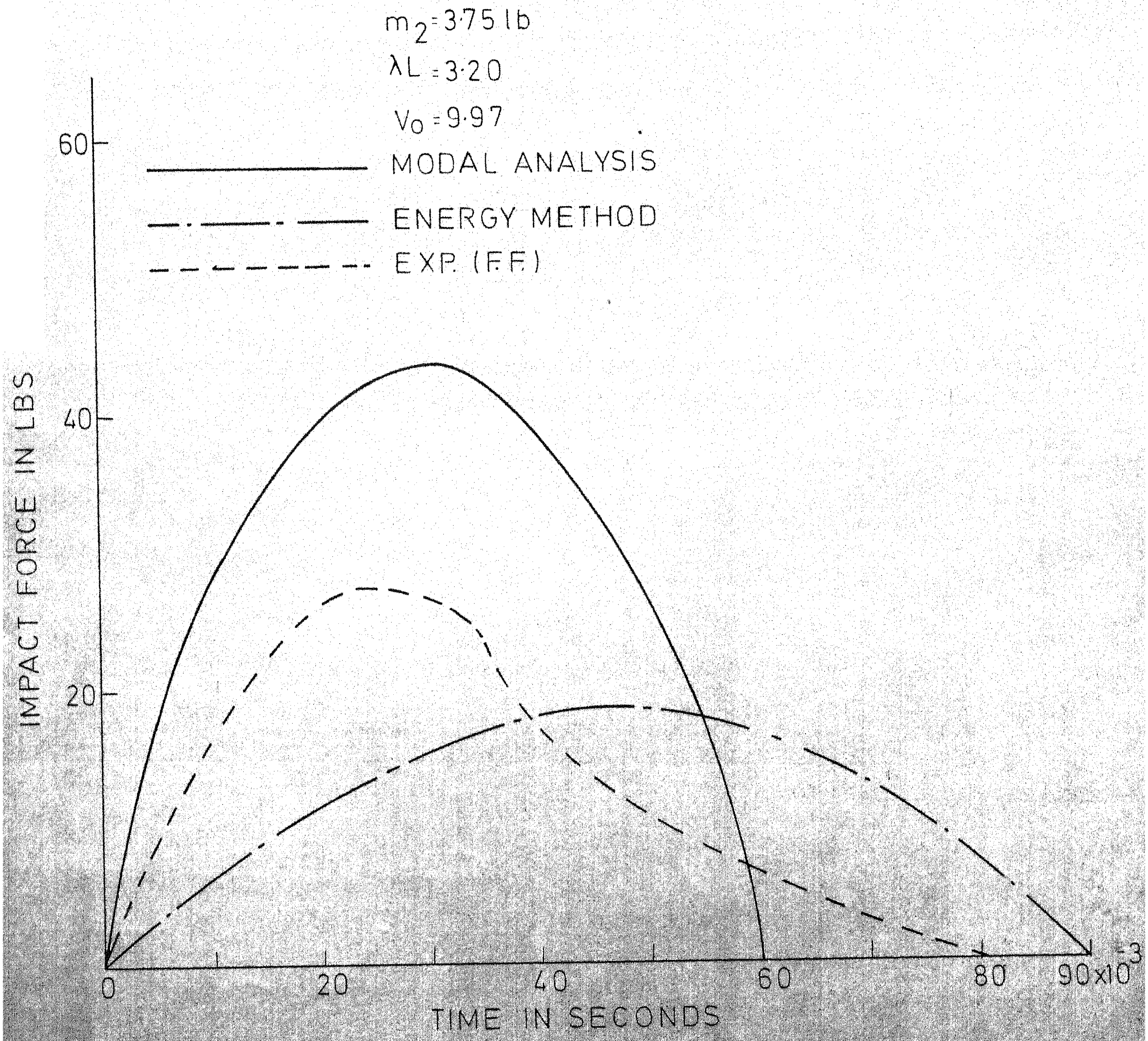


FIG. 2·8 VARIATION OF IMPACT FORCE WITH TIME

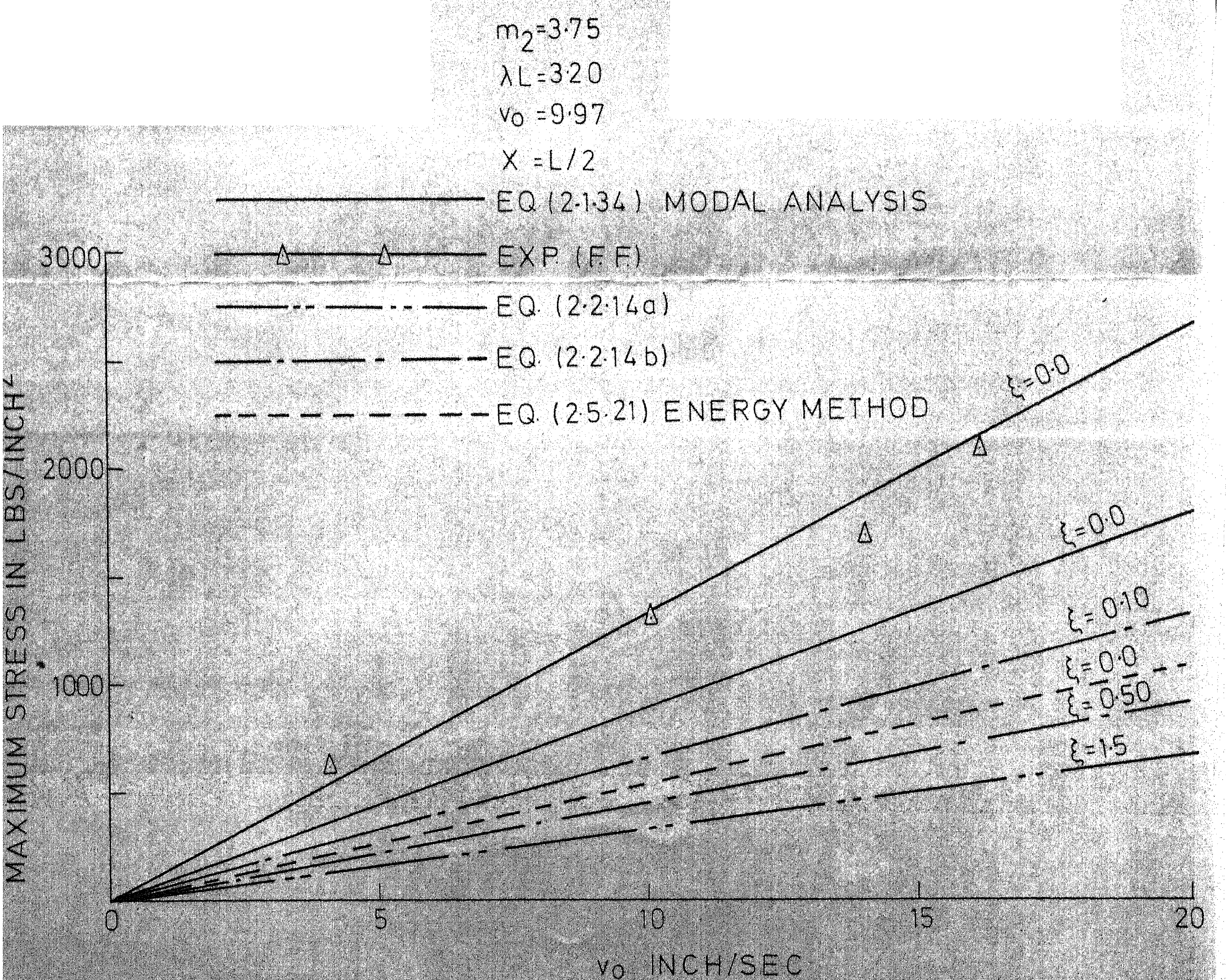


FIG. 2·8a VARIATION OF MAXIMUM FIBRE STRESS WITH v_0

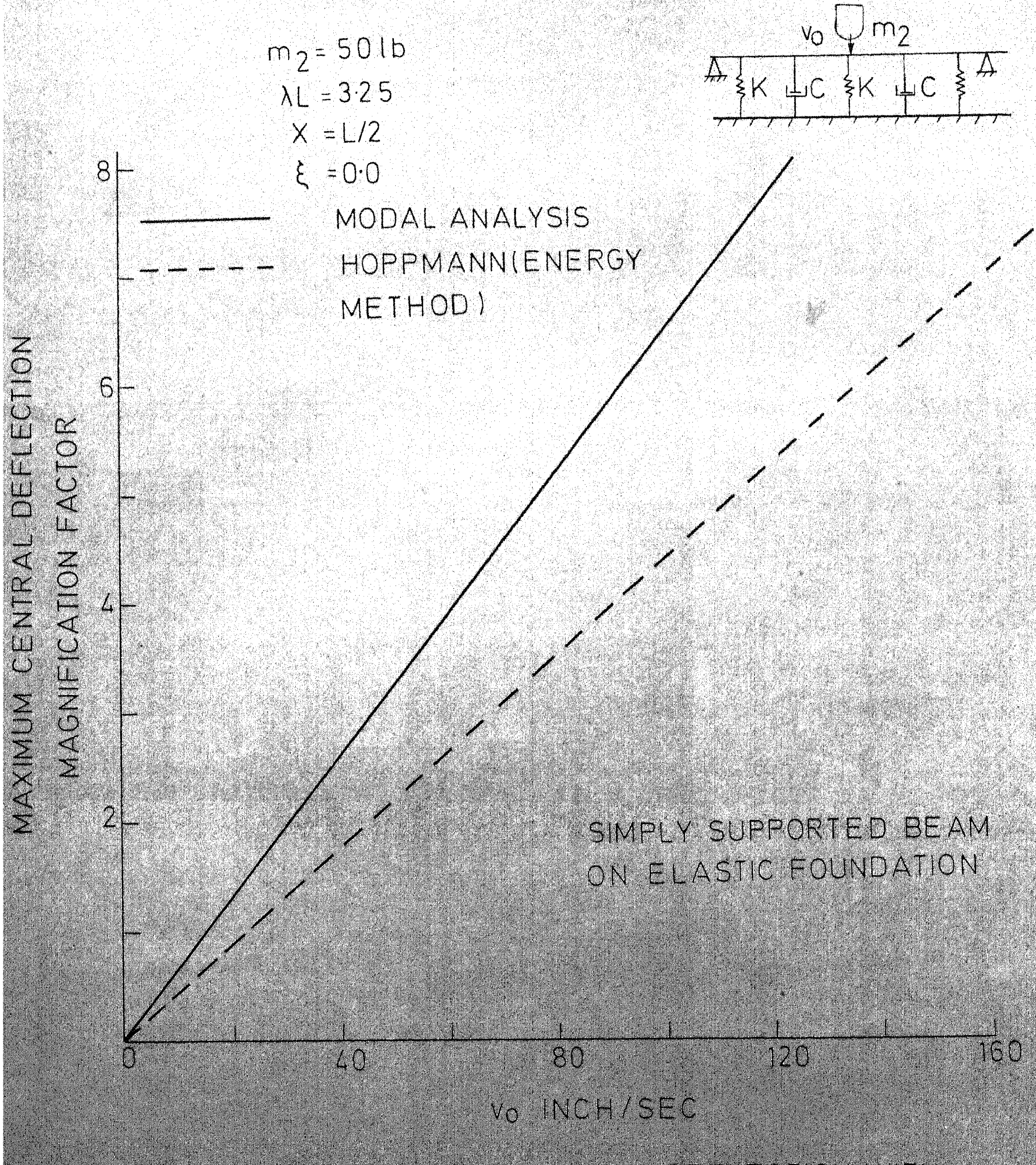


FIG. 2·9 VARIATION OF MAXIMUM DEFLECTION WITH v_0

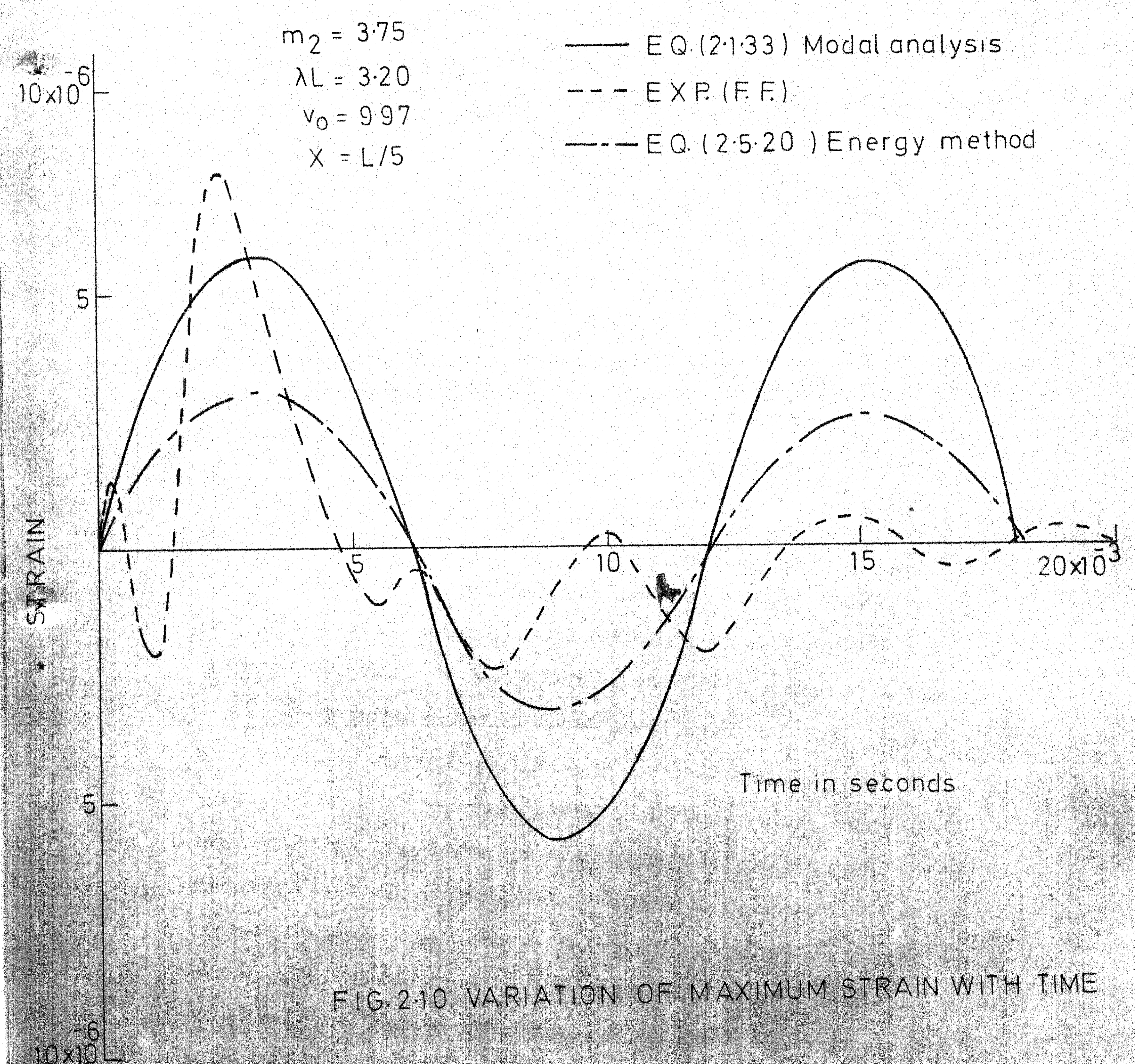


FIG. 2.10 VARIATION OF MAXIMUM STRAIN WITH TIME

CHAPTER 3

DISPLACEMENT BOUNDS FOR BEAMS ON ELASTIC FOUNDATIONS

3.1 INTRODUCTION

From the viewpoint of Engineering Analysis, and design, a principal feature of the response to transient loading or impact loading is the permanent deformation that occurs in the beam on elastic foundation. It is therefore of general engineering interest to have techniques at hand which establish bounds on the overall deformation in dynamic plasticity problems, but which, at the same time allow some or all of the complicating details of the complete response to be ignored [29] .

Even for deformations within the elastic range, the bounds on displacement and energy, is of great help in doing the optimal design of such beams on elastic foundation, whereas it may not be easy to keep these bounds as the objective function and do the conventional optimal design, Namely a special design problem of this kind, may be stated as follows : for a given bound on deflection under specified dynamic loading, what design provides **minimum** total weight ? [1, 16, 38] . The design which minimizes this response bound for dynamic load has been shown to be quite different from the optimal design for minimum deflection under static loading [26] . Here an attempt has been made to compute upperbound on deflection of a beam on elastic foundation by means of an energy approach and some suitable inequalities. The energy has been taken as functional, and for obtaining a bound on

displacement of the structure, an inequality relating the functional to the displacement is required. Since the actual motion is not known, this inequality must hold for all possible displacements. The determination of the optimum inequality of this type is given in section(3.5.) As the objective of this chapter is to find the upper-bounds on deflection of the beam-foundation system at the interface, damping has been neglected (as damping is known to reduce the amplitudes) in the present analysis.

3.2 MARTIN'S APPROACH :

For beams resting on an elastic foundation subjected to dynamic loading it is generally not possible to write a simple analytical expression for the maximum displacement at a given point on the beam. But for a certain class of dynamic loading, it is possible, however to formulate a comparatively simple analytical expression for an upper bound on the maximum displacement at a point. For the elastic structures, such methods have been examined by Martin [19, 20,21,22] . Martin has illustrated the displacement bound principles for elastic and inelastic continuua, which are in fact inequalities relating equilibrium stress fields and compatible displacement or velocity field in the continuum. Treating this beam as a continuum, considering a stress field $_{ij}$ such that

$$\frac{\partial \sigma_{ij}^{s}}{\partial x_j} + F_i = 0 \qquad (3.2.1)$$

at each point in the body and

$$\sigma_{ij}\, a_j = T_i^{s} \qquad (3.2.2)$$

at each point on the surface. The stresses σ_{ij}^{s} are thus in internal equilibrium with body forces F_i and in external equilibrium with surface Traction T_i^{s}, where 'a_j' is a unit outward vector normal to the surface at the point under consideration. Further σ_{ij}^{s} at each point should not violate the initial

yield condition. Then considering a continuous displacement field u_i^c and strain field ε_{ij}^c, such that at each point

$$\varepsilon_{ij}^{c} = \frac{1}{2} \left(\frac{\partial u_i^{c}}{\partial x_j} + \frac{\partial u_j^{c}}{\partial x_i} \right) \qquad (3.2.3)$$

Further it is required that the displacement and strain history following a path from $\varepsilon_{ij} = 0$, $u_i = 0$ to ε_{ij}^{c}, u_i^{c} should be known.

Since σ_{ij}^{s}, T_i^{s}, F_i are in equilibrium and ε_{ij}^{c}, u_i^{c} are compatible, by virtual work principle

$$\int_A T_i^{s} u_i^{c} \, dA + \int_V F_i u_i^{c} \, dV = \int_V \sigma_{ij}^{s} \varepsilon_{ij}^{c} \, dV \qquad (3.2.4)$$

For the dynamic problems under consideration, the conditions are:

1. At time $t < t_o$ body forces F_i act on the continuum, causing displacements and strains, u_i', ε_{ij}' such that all stresses in this state are within yield condition.

2. At time $t = t_o$, the continuum accquires velocity $\dot{u}_i^{o}$.

3. For time $t > t_o$, the surface tractions are zero over part of the surface and the displacement rates are zero over the remainder of the surface.

4. For $t > t_o$ the displacement, velocities and strains are given by u_i^t, $\dot{u}_i^t$, ε_{ij}^t.

Then the energy balance equation relating the states at t_o and t, may be written as

$$\int_V dV \int_0^{\varepsilon_{ij}'} \sigma_{ij} d\varepsilon_{ij} + \int_V \frac{\rho}{2} \dot{u}_i^o \dot{u}_i^o dV = \int_V dV \int_0^{\varepsilon_{ij}^t} \sigma_{ij} d\varepsilon_{ij}$$

$$+ \int_V \frac{\rho}{2} \dot{u}_i^t \dot{u}_i^t dV - \int F_i (u_i^t - u_i') dV \qquad (3.2.5)$$

which finally reduces to

$$\int_V dV \int_0^{\sigma_{ij}^s} \varepsilon_{ij} d\sigma_{ij} + \int_V \frac{\rho}{2} \dot{u}_i^o \dot{u}_i^o dV \geq \int_A T_i^s u_i^t dA \qquad (3.2.6)$$

Thus for the material obeying the Drucker's postulate of stability(3,4,5), the principle may be written as

$$\int_A T_i^s u_i^t dA \leq \int_V \frac{\rho}{2} \dot{u}_i^o \dot{u}_i^o dV + \int_V \varepsilon_{ij}(\sigma_{ij}^s) dV \qquad (3.2.7)$$

where T_i^s = Any safe set of surface tractions

$\int_V \varepsilon_{ij} \sigma_{ij}^s dV = C^s$ = Complementary energy associated with stress field σ_{ij}^s, which does not violate yield condition.

a) SIMPLY SUPPORTED BEAM ON ELASTIC FOUNDATION :

A uniform simply supported beam of elastic material resting on a Winkler model of foundation having a foundation constant K, is subjected to impact loading such that initial velocity distribution is known as shown in fig. (3.1b). The moment curvature relationship of the beam is as shown in fig. (3.1c), treating the material elastic and perfectly plastic which also satisfies the stability postulate [3,4,5].

If the central transverse displacement of the dynamically loaded beam [figure (3.1a,b)] at any time during the impulse is given by δ, then to satisfy Eq.(3.2.7)

$$P.\delta \leq K^{o} + C^{s} \qquad (3.2.8)$$

neglecting all other terms including shear deformation.

Where

K^{o} = Initial Energy terms (body forces not included) for initial velocity distribution of $v(x)$

$$= 2 \int_{0}^{L/2} \tfrac{1}{2}.\rho A.\ v(x).\ v(x)\,dx \qquad (3.2.9)$$

which is analogus to term $\int_{V} \frac{1}{2}.\rho \dot{u}_i^{o} . \dot{u}_i^{o} . dV$ in Eq.(3.2.7).

C^{s} = Complementary strain energy stored in the beam and foundation

P represent the choice of T_i^{s} as shown in Fig. (3.1a).

So $$C^s = \frac{1}{2EI} \int_0^L \left(\frac{d^2W}{dx^2} \right)^2 dx + \frac{1}{2} K \int_0^L W^2 dx \qquad (3.2.10)$$

which is analogus to $\int_{\forall} \varepsilon_{ij}(\sigma_{ij}^{s}) d\forall$ in Eq. (3.2.7).

Now considering any specific value of P, the optimum bound can be obtained by maximizing δ, such that

$$\frac{\partial \delta}{\partial P} = 0 \qquad (3.2.11)$$

For this simply supported beam $0 \leq x \leq L$, in a statically admissible state, with a central load P,

$$\text{Bending Moment} = M_x = \frac{P}{4\lambda} \quad C_{\lambda x} \qquad (3.2.12)$$

$$\text{Deflection} = W = \frac{P\lambda}{2K} \quad A_{\lambda x} \qquad (3.2.13)$$

$$\text{Maximum Bending Moment at centre} = \frac{P}{4\lambda} \cdot \phi \qquad (3.2.14)$$

where,

$$C_{\lambda x} = \frac{1}{\mathrm{Cosh}\,\lambda L + \mathrm{Cos}\,\lambda L} \{ \mathrm{Cos}\,\lambda x\ \mathrm{Sin}\,\lambda(L-x) - \mathrm{Sinh}\,\lambda x\ \mathrm{Cos}\,\lambda(L-x) - \mathrm{Sin}\,\lambda x\ \mathrm{Cosh}\,\lambda(L-x) + \mathrm{Cos}\,\lambda x\ \mathrm{Sinh}\,\lambda(L-x) \} \qquad (3.2.15)$$

$$A_{\lambda x} = \frac{1}{\mathrm{Cosh}\,\lambda L + \mathrm{Cos}\,\lambda L} \{ \mathrm{Cos}\,\lambda x\ \mathrm{Sinh}\,\lambda(L-x) - \mathrm{Cosh}\,\lambda x\ \mathrm{Sin}\,\lambda(L-x) + \mathrm{Sin}\,\lambda x\ \mathrm{Cos}\,\lambda(L-x) - \mathrm{Sinh}\,\lambda x\ \mathrm{Cos}\,\lambda(L-x) \} \qquad (3.2.16a)$$

$$\phi = \frac{\text{Sinh } \lambda L + \text{ Sin } \lambda L}{\text{Cosh } \lambda L + \text{ Cos } \lambda L} \qquad (3.2.16b)$$

in which $\lambda = [K/4EI]^{1/4}$ (3.2.17)

So strain energy stored in the beam and foundation

$$C^s = \frac{P^2}{128 \text{ EI}} (\phi_1 + \psi_1) \qquad (3.2.18)$$

where $\phi_1 = \frac{4}{\lambda^2} \int_0^L (C_{\lambda x})^2 \, dx$ (3.2.19)

$$\psi_1 = \frac{1}{\lambda^2} \int_0^L (A_{\lambda x})^2 \, dx \qquad (3.2.20)$$

However, this complementary strain energy can be more easily computed considering the desired solution Eq.(3.2.13) in the form of trigonometric series for the beam of finite length. Hateyni [9] has illustrated that a Sine series of the form

$$W = \sum_{n=1}^{\infty} a_n . \frac{\text{Sin } n\pi x}{L} \qquad (3.2.21)$$

will satisfy completely the boundary condition at both ends.

Strain energy of bending of the beam

$$V_1 = C^s_{\text{BEAM}} = \frac{\pi^4 EI}{4L^3} \sum_{n=1}^{\infty} n^4 \, a_n^2 \qquad (3.2.22)$$

and that of foundation

$$V_2 = C_{Foundation} = \frac{KL}{4} \sum_{n=1}^{\infty} a_n^2$$

For small variation in a_n, the resulting change in strain energy will be

$$\frac{\partial V}{\partial a_n} da_n = \frac{\partial V_1}{\partial a_n} da_n + \frac{\partial V_2}{\partial a_n} da_n$$

$$= \frac{\pi^4 EI}{2L^3} n^4 a_n da_n + \frac{KL}{2} a_n da_n$$

At the same time, because of the change da_n, the work done by loading P, at a distance L/2, from the support, is given by

$$P \frac{\partial}{\partial a_n} W \, d a_n = P \operatorname{Sin} \frac{n\pi c}{L} d a_n \qquad (3.2.23)$$

Equating this change in potential energy to the change in the strain energy of the system

$$P \sin \frac{n\pi c}{L} = \frac{\pi^4 EI}{2L^3} n^4 a_n + \frac{KL}{2} a_n$$

where $c = L/2$.

So 'a_n' becomes,

$$a_n = \frac{2PL^3}{\pi^4 EI} \frac{\operatorname{Sin} \frac{n\pi}{2}}{n^4 + \frac{KL^4}{\pi^4 EI}} \qquad (3.2.24)$$

Hence $\quad C^s = \frac{P^2}{128EI} (\phi_1 + \psi_1)$

where

$$\phi_1 = \sum_{n=1}^{\infty} \frac{128}{\pi^4} n^4 L^3 \left[\frac{\text{Sin } n\pi/2}{n^4+4(\lambda L/\pi)^4} \right]^2 \tag{3.2.25}$$

$$\psi_1 = \sum_{n=1}^{\infty} \frac{128}{\pi^4} (\lambda L)^4 L^3 \left[\frac{\text{Sin } n\pi/2}{n^4 + 4(\lambda L/\pi)^4} \right]^2 \tag{3.2.26}$$

and $$W = \frac{2PL^3}{\pi^4 EI} \sum_{n=1}^{\infty} \left[\frac{\text{Sin } n\pi/2 \text{ Sin } n\pi x/L}{n^4 + 4(\lambda L/\pi)^4} \right] \tag{3.2.27}$$

Considering a state when bending moment anywhere in the beam is less than the limiting moment M_o,

$$\frac{P}{4\lambda} \cdot \phi \leq M_o \tag{3.2.28}$$

As long as Eq.(3.2.28) is satisfied, the entire beam is in elastic range. Now optimizing 'δ' with respect to P, in Eq.(3.2.8),

$$\frac{\partial}{\partial P}(P.\delta) = \frac{\partial}{\partial P}(K^o + C^s)$$

$$P\frac{\partial\delta}{\partial P} + \delta = \frac{\partial C^s}{\partial P}$$

$$\delta = \frac{\partial C^s}{\partial P} = \frac{K^o + C^s}{P} \tag{3.2.29}$$

putting the value of C^s from Eq.(3.2.18)

$$P = \left(\frac{128EI\ K^o}{\phi_1 + \psi_1} \right)^{\frac{1}{2}} \tag{3.2.30}$$

Thus value of P, is given by Eq.(3.2.30) as long as $M_x \leq M_o$

So $$P \leq \frac{4 \lambda M_o}{\phi} \tag{3.2.31}$$

i.e., $$\left(\frac{128EI\ K^o}{\phi_1 + \psi_1}\right)^{\frac{1}{2}} \leq \frac{4 \lambda M_o}{\phi} \tag{3.2.32}$$

So when the response is elastic for the entire beam , the value of P given by Eq.(3.2.30) should be adopted and otherwise when Eq.(3.2.31) is not satisfied, the value of P given by the equality in Eq. (3.2.28) should be taken. Thus when the entire response is elastic

$$\delta \leq \frac{K^o + C^s}{P}$$

$$\delta \leq \left[\frac{K^o(\phi_1+\psi_1)}{32\ EI}\right]^{\frac{1}{2}} \quad \text{for } M_o \geq \left[\frac{8EI\ K^o\ \phi^2}{\lambda^2(\phi_1 + \psi_1)}\right]^{\frac{1}{2}} \tag{3.2.33}$$

On the otherhand if

$$M_o \leq \left[\frac{8EI\ K^o\ \phi^2}{\lambda^2(\phi_1 + \psi_1)}\right]^{\frac{1}{2}}$$

Taking the value of P from (3.2.28),

$$P = \frac{4\lambda\ M_o}{\phi}$$

Then $$\delta \leq \frac{K^o\ \phi}{4\lambda\ M_o} + \frac{M_o \lambda\ (\phi_1 + \psi_1)}{32\ EI\ \phi} \tag{3.2.34}$$

Eq.(3.2.33) and Eq.(3.2.34) gives the general solution for a given value of initial kinetic energy input. As an illustration for comparison of results taking a half sine curve for initial velocity distribution [Ref. 19] , given by

$$v(x) = v_o \operatorname{Sin} \frac{\pi x}{L} \tag{3.2.35}$$

$$K^o = \int_o^L \frac{m}{2} v_o^2 \operatorname{Sin}^2 \frac{\pi x}{L} dx$$

where $m = \rho A$ = mass of beam per unit length.

Introducing a dimensionless impulse parameter

$$S = \frac{mEI\ v_o^2}{2M_o^2} (\phi/\lambda L)^2 \frac{L^3}{\phi_1 + \psi_1} \tag{3.2.36}$$

and a dimensionless displacement parameter,

$$D = \frac{M_o \cdot \delta}{mL^2\ v_o^2} \tag{3.2.37}$$

Then equation (3.2.33) reduces to

$$\frac{M_o\ \delta}{mL^2 v_o^2} \leq \frac{1}{16}(S)^{\frac{1}{2}} (\phi/\lambda L) \text{ for } S \leq \tfrac{1}{4} \tag{3.2.38}$$

and equation (3.2.34) reduces to

$$\frac{M_o \cdot \delta}{mL^2 v_o^2} \leq \left(\frac{1}{16} + \frac{1}{64S}\right) \frac{\phi}{\lambda L} \text{ for } S \geq \tfrac{1}{4} \tag{3.2.39}$$

The values of S,D, for $\lambda L = 1.0$ is given in Table (3.1). And S and D have been plotted on a semi-log scale for different values of λL in fig. (3.2).

3.3 FREE-FREE BEAM ON ELASTIC FOUNDATION:

The beam $0 \leq x \leq L$ with load P fig. (3.1d) in statically admissible state, has the following characteristics

$$\text{B.M. at any distance } x = M_x = \frac{P}{4\lambda} C_{\lambda x} \qquad (3.3.1)$$

$$\text{Deflection} = W = \frac{P\lambda}{2K} A_{\lambda x} \qquad (3.3.2)$$

$$\text{Maximum B.M. at centre} = (P/4\lambda)\phi \qquad (3.3.3)$$

$$\begin{array}{l}\text{Maximum Deflection}\\ \text{at the Centre}\end{array} = \frac{P\lambda}{2K} \frac{\cosh\lambda L + \cos\lambda L + 2}{\sinh\lambda L + \sin\lambda L} \qquad (3.3.4)$$

H where

$$\begin{aligned} C_{\lambda x} = \frac{1}{\sinh\lambda L + \sin\lambda L} \{ & \sinh\lambda x\,[\sin\lambda x - \sin\lambda(L-x)] - \\ & \cosh\lambda x\,[\cos\lambda x + \cos\lambda(L-x)] + \\ & \sin\lambda x\,[\sinh\lambda x - \sinh\lambda(L-x)] + \\ & \cos\lambda x\,[\cosh\lambda x + \cosh\lambda(L-x)] \} \end{aligned} \qquad (3.3.5)$$

$$\begin{aligned} A_{\lambda x} = \frac{1}{\sinh\lambda L + \sin\lambda L} \{ & \cosh\lambda x \cos\lambda(L-x) + \\ & \cos\lambda x \cosh\lambda(L-x) - \\ & \sinh\lambda x \sin\lambda(L-x) + \\ & \sin\lambda x \sinh\lambda(L-x) + \\ & 2\cosh\lambda x \cos\lambda x \} \end{aligned} \qquad (3.3.6)$$

$$\phi = \frac{\text{Cosh } \lambda L - \text{Cos } \lambda L}{\text{Sinh } \lambda L + \text{Sin } \lambda L} \tag{3.3.7}$$

So with free-free boundary conditions, the only change from simply supported Beam Analysis, will be that the C^S will be given by

$$C^S = \frac{P^2}{128EI} (\phi_1 + \psi_1) \tag{3.3.8}$$

where
$$\phi_1 = \frac{4}{\lambda^2} \int_0^L (C_{\lambda x})^2 \, dx \tag{3.3.9}$$

$$\psi_1 = \frac{1}{\lambda^2} \int_0^L (A_{\lambda x})^2 \, dx \tag{3.3.10}$$

And $C_{\lambda x}$ and $A_{\lambda x}$ are given by Eq.(3.3.5) and (3.3.6).

Following the analysis presented in the previous section (Sec. 3.2), the final equations governing the deflection can be obtained as

$$\frac{M_o \cdot \delta}{mL^2 v_o{}^2} \leq \frac{1}{16} (1/S)^{\frac{1}{2}} (\phi/\lambda L) \text{ for } S \leq \tfrac{1}{4} \tag{3.3.11}$$

$$\frac{M_o \cdot \delta}{mL^2 v_o^2} \leq \left(\frac{1}{16} + \frac{1}{64S}\right) \frac{\phi}{\lambda L} \quad \text{for } S \geq \tfrac{1}{4} \tag{3.3.12}$$

The value of S and D for $\lambda L = 1.0$ is given in Table (3. 2). S and D have been plotted on a semi-log scale for different value of λL in Fig. (3.3). The values of D for free-free case are much less than for simply supported case.

3.4 ILLUSTRATED EXAMPLE :

(a) As an example of the application of this principle of displacements computation, a simply supported Beam on Elastic foundation subjected to impulse loading is considered and the computed bounds are compared with the exact solution obtained earlier (Sec. 2.1) using Modal Analysis. The accuracy of computed bounds is reflected in this comparision. But for a very simple case with well defined impulse, exact solution are available and so this comparision does not necessarily reflect the accuracy of these bounds for different boundary and initial conditions.

Considering a uniform simply supported beam on elastic foundation with the initial condition,

$$W(x,0) = 0$$

$$\frac{\partial W}{\partial t}(x,0) = v_o \frac{\operatorname{Sin} \pi x}{L} \qquad (3.4.1)$$

The governing differential equation is

$$EI \frac{\partial^4 W(x,t)}{\partial x^4} + m \frac{\partial^2 W(x,t)}{\partial t^2} + KW(x,t) = 0$$

has the solution,

$$W(x,t) = \sum_{n=1}^{\infty} \operatorname{Sin} \frac{n\pi x}{L} \left(A_n \operatorname{Cos} w_n t + B_n \operatorname{Sin} w_n t\right) \qquad (3.4.2)$$

Using the initial conditions given by Eqns.(3.4.1)

$$W(x,t) = \frac{v_o}{w_1} \ \mathrm{Sin}\,\frac{\pi x}{L}\ \mathrm{Sin}\ w_1 t \qquad (3.4.3)$$

Thus showing that the beam will vibrate only in first mode where the first natural frequency is given by

$$w_1 = \frac{\pi^2}{L^2}\left[1 + \frac{KL^4}{EI\,\pi^4}\right]^{1/2}\left[\frac{EI}{m}\right]^{1/2}$$

$$\lambda = \left[\frac{K}{4EI}\right]^{1/4}$$

Eq. (3.4.3) becomes

$$W(x,t) = \frac{v_o L^2}{\pi^2}\ \mathrm{Sin}\ \frac{\pi x}{L}\ \frac{\mathrm{Sin}\ w_1 t}{\sqrt{1+4(\lambda L/\pi)^4}} \cdot \left[\frac{m}{EI}\right]^{1/2}$$

For maximum central deflection

$$\mathrm{Sin}\ \frac{\pi x}{L}\Bigg|_{x = L/2} = 1.0$$

Allowing this to happen at time t', such that

$$\mathrm{Sin}\ w_1 t' = 1$$

$$W_{max} = \frac{v_o L^2}{\pi^2}\ \frac{1}{[1+4(\lambda L/\pi)^4]^{1/2}} \cdot \left[\frac{m}{EI}\right]^{1/2}$$

For $\lambda L = \pi$.

$$W_{(maximum)} = 0.04550\, v_o L^2 . \left[\frac{m}{EI} \right]^{1/2} \qquad (3.4.4)$$

Using Eq.(3.1.33) to evaluate the bound for central deflection,

$$\delta = \left[\frac{K^o(\phi_1 + \psi_1)}{32EI} \right]^{\frac{1}{2}} \qquad (3.4.5)$$

where $$K^o = \frac{mLv_o^2}{4}$$

ϕ_1 and ψ_1 are obtained from Eqs. (3.1.25) and (3.1.26) for $\lambda L = \pi$, taking first three terms only, as the series is rapidly converging and are obtained as

$$\phi_1 + \psi_1 = 0.2572\ L^3 \qquad (3.4.6)$$

Hence from Eqns. (3.4.5) and (3.4.6)

$$\delta = v_o . L^2 .\ 0.0515 \left[\frac{m}{EI} \right]^{1/2} \qquad (3.4.5)$$

Comparing this with exact solution given by Eqn. (3.4.4) the computed bound is only 12% higher than the exact value.

(b) Considering simply supported beam, resting on an elastic foundation, which is subjected to a central impact with a mass m_2, and velocity v_o. Assuming no loss of energy, the kinetic energy input is taken as

$$K^o = \frac{1}{2} \quad m_2 v_o^2$$

Using Eqn.(3.2.33), upper bound on displacement is

$$\delta_{max} = [\frac{K^{o}(\phi_1 + \psi_1)}{32EI}]^{\frac{1}{2}}$$

Hence

$$\delta_{max.} = \frac{1}{2} [\frac{m_2 \ \phi_1 + \psi_1}{32 \ EI}]^{\frac{1}{2}} \cdot v_o = \frac{1}{8} v_o [\frac{m_2 \cdot (\phi_1 + \psi_1)}{EI}]^{\frac{1}{2}}$$

Taking $\lambda L = \pi$, $\phi_1 + \psi_1 = 0.2572 \ L^3$

$$\delta_{max.} = \frac{1}{8} v_o [\frac{m_2 \ x \ 0.2572}{EI}]^{\frac{1}{2}} \cdot L^{3/2} .$$

For v_o = 100 inch/sec., m_2 = 3.76 , EI = 6×10^5 lb-inch^2, L = 40
e = 0.22 δ_{max} = 0.101 inch.

The maximum displacement by modal analysis

$$W_{max} = 0.077 \text{ inch}$$

Thus the computed bound is 1.30 times more than the maximum value given by Modal Analysis.

TABLE 3.1

FOR SIMPLY SUPPORTED BEAM
ON ELASTIC
FOUNDATION

λL	ϕ	$\phi/\lambda L$
0.5	0.60	1.20
1.0	1.00	1.00
2.0	1.35	0.675
π	1.09	0.348
4.0	0.99	0.248
5.0	0.982	0.196
2π	1.001	0.158
8.0	1.01	0.125

For $\lambda L = 1.0$, Value of S and D

S	D	REMARKS
0.10	0.187	ELASTIC RANGE
0.20	0.140	
0.25	0.125	
0.50	0.0938	
1.00	0.0782	
2.00	0.0704	
5.00	0.0657	
10.00	0.0641	
100.00	0.0627	
.		
.		
.		
∞	0.0626	

TABLE 3.2

FOR FREE FREE BEAM ON ELASTIC FOUNDATION

λL	ϕ	$\phi/\lambda L$
0.50	0.252	0.504
1.00	0.505	0.505
2.00	0.924	0.464
π	1.09	0.347
2π	0.995	0.160
8.0	0.98	0.124

For $\lambda L = 1.0$, Value of S and D

S	D	REMARKS
0.10	0.099	ELASTIC
0.20	0.070	RANGE
0.25	0.062	
0.50	0.0470	
1.00	0.0390	
2.00	0.0350	
5.00	0.0325	
10.00	0.0320	
100.00	0.0312	
.		
.		
.		
∞	0.0310	

3.5. BOUNDS ON DISPLACEMENT AND ENERGY USING VARIATIONAL PRINCIPLE.

An attempt is made to evaluate the upper bound on energy and displacement for beams on elastic foundations subjected to a general class of dynamic loading $q(x,t)$. The governing differential equation of motion is (neglecting the damping)

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 W}{\partial x^2}\right] + m(x)\frac{\partial^2 W}{\partial t^2} + K(x)\,W = q(x,t) \qquad (3.5.1)$$

Defining $EI(x) = S(x)$ = Bending Stiffness.

With initial conditions as

$$W(0,t) = 0 \quad \text{and} \quad \frac{\partial W}{\partial t}(x,0) = 0 \qquad (3.5.2)$$

The following end conditions of the beam are

(a) Pinned - Pinned

$$\left.\begin{aligned} &W(0,t) = 0\,, \quad S(0)\frac{\partial^2 W(0,t)}{\partial x^2} = 0 \\ &W(L,t) = 0, \quad S(L)\frac{\partial^2 W(L,t)}{\partial x^2} = 0 \end{aligned}\right| \qquad (3.5.3)$$

(b) Clamped - Clamped

$$\left.\begin{aligned} &W(0,t) = 0\,, \quad \frac{\partial W}{\partial x}(0,t) = 0 \\ &W(L,t) = 0\,, \quad \frac{\partial W}{\partial x}(L,t) = 0 \end{aligned}\right| \qquad (3.5.4)$$

(c) Clamped - Free

$$\left.\begin{aligned} & W(0,t) = 0 \;,\; \frac{\partial W(0,t)}{\partial x} = 0 \\ & S(L)\,\frac{\partial^2 W(L,t)}{\partial x^2} = 0 \;,\; \frac{\partial}{\partial x}\left[S(L)\,\frac{\partial^2 W(L,t)}{\partial x^2}\right] = 0 \end{aligned}\right| \qquad (3.5.5)$$

(d) Free- Free

$$\left.\begin{aligned} & S(0)\,\frac{\partial^2 W(0,t)}{\partial x^2} = 0 \;,\; \frac{\partial}{\partial x}\left[S(0)\,\frac{\partial^2 W(0,t)}{\partial x^2}\right] = 0 \\ & S(L)\,\frac{\partial^2 W(L,t)}{\partial x^2} = 0 \;,\; \frac{\partial}{\partial x}\left[S(L)\,\frac{\partial^2 W(L,t)}{\partial x^2}\right] = 0 \end{aligned}\right| \qquad (3.5.6)$$

Considering an elastic beam resting on elastic foundation (Winkler Model) with one of the above mentioned boundary conditions, 'x' is the axial distance along the beam, $x = 0$, being the one end and $x = L$ the other end, 't' is the time parameter and $W(x,t)$ is the lateral displacement. The bending stiffness is denoted by $EI(x) = S(x)$ and the mass per unit length by $m(x)$. $S(x)$, $m(x)$ assumed to be symmetric about $x = L/2$.

If the dynamic load $q(x,t)$ is applied to the beam at $t = 0$, the equation of motion is given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\,\frac{\partial^2 W(x,t)}{\partial x^2}\right] + m(x)\,\frac{\partial^2 W(x,t)}{\partial x^2} + KW(x,t) = q(x,t) \qquad (3.5.7)$$

Neglecting shear deformation and rotatory inertia.defining the function

$$V^* = 2[\text{Strain Energy} + \text{Kinetic Energy}] \tag{3.5.8}$$

$$V^* = \int_0^L [S(x)\ (\frac{\partial^2 W}{\partial x^2})^2 + KW^2 + m(x)\ (\frac{\partial W}{\partial t})^2]\ dx \tag{3.5.9}$$

The time rate of change of V*during motion of the beam is given by

$$\frac{dV^*}{dt} = 2 \int_0^L [S(x)\ \frac{\partial^2 W}{\partial x^2} \frac{\partial^3 W}{\partial x^2 \partial t} + KW \frac{\partial W}{\partial t} + m(x)\ \frac{\partial W}{\partial t} \frac{\partial^2 W}{\partial t^2}]\ dx$$

Integrating by parts, together with boundary condition, the above equation can be expressed as

$$\frac{dV^*}{dt} = 2 \int_0^L [\frac{\partial^2}{\partial x^2} (S(x)\ \frac{\partial^2 W}{\partial x^2})] + KW + m(x)\ \frac{\partial^2 W}{\partial t^2}]\ \frac{\partial W}{\partial t} \cdot dx$$

With the use of equation of motion given by Eq. (3.5.7)

$$\frac{dV^*}{dt} = 2 \int_0^L q(x,t)\ \frac{\partial W}{\partial t}\ dx \tag{3.5.10}$$

It follows from Schwarz inequality [11] and definition of V* (Eq. 3.5.9) that

$$\frac{dV^*}{dt} \leq 2\ \alpha(t)\ \{\int_0^L m(x)\ (\frac{\partial W}{\partial t})^2\ dx\ \}^{1/2} \leq 2\ \alpha(t)\ V^{*\frac{1}{2}}$$

Defining $V^* = V_1^2$ where $V_1 \geq 0$

$$\frac{dV^*}{dt} = 2V_1 \frac{dV_1}{dt}$$

$$2V_1 \frac{dV_1}{dt} \leq 2\ \alpha(t)\ .\ V_1$$

$$\frac{dV_1}{dt} \leq \alpha(t) \tag{3.5.10a}$$

Integrating Eq. (3.5.10a) and using the initial conditions Eq. (3.5.2)

$$V_1 \leq \int_0^t \alpha(\tau)\ d\ \tau$$

$$V^{*\frac{1}{2}} \leq \int_0^t \alpha(\tau)\ d\tau \tag{3.5.11}$$

or

$$\{\int_0^L [S(x)\ (\frac{\partial^2 W}{\partial x^2})^2 + m(x)\ (\frac{\partial W}{\partial t})^2 + KW^2]\ dx\}^{\frac{1}{2}} \leq \int_0^t \alpha(\tau)\ d\tau \tag{3.5.12}$$

This inequality furnishes an upper bound on energy.

For obtaining a bound on displacement, it is sufficient to derive an inequality of the form [26,27]

$$\int_0^L (\frac{\partial^2 W}{\partial x^2})^2\ dx \geq B^2\ W_m^2 \tag{3.5.13}$$

Where W_m = Maximum $|W|$ for $0 \leq x \leq L$

and B = constant

or $\int_0^L (W_{xx})^2 \, dx \geq B^2 \, W_m^2$

Equation (3.5.13) is to be satisfied at time 't' by all possible displacements W. In addition the largest value of constant B in Eq. (3.5.13) is to be obtained.

The value of 'B' is derived by the procedure suggested by Plant [26]. As stated above, the above inequality (Eq. 3.5.13) is to hold for any time t, the problem of determining the extremum value of constant B; can be reduced to a time-independant variational problem [26,27].

To solve the variational problem, an auxiliary function u(x) is introduced. The function u(x) will be called admissible if it satisfies the appropriate boundary conditions for the beam, resting on elastic foundation and if u(x) is continuous and if

$$\int_0^L (u_{xx})^2 \, dx < \infty \qquad (3.5.14)$$

Considering all admissible deflections for which

$$\int_0^L (u_{xx})^2 \, dx = C^2 \qquad (3.5.15)$$

relation given by Eq. (3.5.15) may be viewed as a normalisation condition on admissible deflections. The largest value of 'B' is then determined in two steps. First, the particular admissible function which satisfies Eq. (3.5.15) and which yields the maximum deflection at some arbitrary point along a beam on elastic foundation,

say x = a is found. Then, the particular value of 'a' say $a = a^*$ at which the foregoing set of maxima has an absolute maximum, is determined. Once the location a^* (therefore the maximum deflection), is known, the extremum value of B, may be calculated.

In mathematical terms, the first step is to maximize the function u(a) for any point $0 \leq a \leq L$, subject to subsidiary condition given by Eq. (3.5.15).

The problem is clearly suited for application of Lagrange's method of undetermined multipliers. The Lagrange's function can be written as

$$F(u) = u(a) + \lambda^* \{\int_0^L (u_{xx})^2 \, dx - C^2\} \qquad (3.5.16)$$

In Eq. (3.5.16), u has to be determined such that F(u) is stationary, where λ^* is a undetermined multiplier.

Using these, the final expression for u(x) can be written [26,27] as

$$u(x) = \frac{C}{\sqrt{g(a|a)}} \, g(x|a) \qquad (3.5.17)$$

Where $g(x|a)$, is the influence function for the beam, resting on elastic foundation and λ^* has been determined in terms of C from Eq. (3.5.15).

The influence function for the beam, resting on the elastic foundation is defined in the customary fashion, as the displacement at 'x' due to a unit load at 'a'. From Eq. (3.5.17) it can be concluded that for a given strain energy, the maximum deflection at $x = a$ is achieved by applying a concentrated force of appropriate magnitude at that point.

The second step is to determine the value of a^* at which the magnitude of $u(a)$ and hence $g(a|a)$ is maximum.

The point a^* is either an end point or an interior point which satisfies

$$\frac{dg(a|a)}{da} = \left[\frac{\partial g(a|a)}{\partial x} + \frac{\partial g(x|a)}{\partial a}\right]_{x=a} = 0 \qquad (3.5.18)$$

Using Maxwell's Reciprocal theorem $g(x|a) = g(a|x)$, Eq. (3.5.18) can also be written as

$$\left[\frac{\partial g(x|a)}{\partial x}\right]_{x=a} = 0 \qquad (3.5.19)$$

Which implies that u_x is zero at the point of application of the load. After solving for a^*, the maximum deflection at any point x can be written as

$$u(x) = \frac{C}{\sqrt{g(a^*|a^*)}} g(x|a^*) \qquad (3.5.20)$$

and the absolute maximum itself as

$$u_m = C\sqrt{g(a^*|a^*)} \tag{3.5.21}$$

The extremum value of B is then given by

$$B^2 = \frac{\int_0^L (u_{xx})^2\,dx}{u_m} = \frac{1}{g(a^*|a^*)} \tag{3.5.22}$$

The quantity $g(a^*|a^*)$ is the maximum possible displacement of the beam, resting on elastic foundation under a unit concentrated load. For

(a) Hinged-Hinged ends

$$a^* = L/2$$

$$g(a^*|a^*) = \frac{\lambda}{2K}\,\frac{\text{Sinh }\lambda L - \text{Sin }\lambda L}{\text{Cosh }\lambda L + \text{Cos }\lambda L} \tag{3.5.23}$$

Where $\lambda = [\frac{K}{4EI}]^{1/4}$.

(b) Fixed-Fixed ends

$$a^* = L/2$$

$$g(a^*|a^*) = \frac{\lambda}{2K}\,\frac{\text{Cosh }\lambda L + \text{Cos }\lambda L - 2}{\text{Sinh }\lambda L + \text{Sin }\lambda L} \tag{3.5.24}$$

(c) Fixed-Free ends

$$a^* = L$$

$$g(a^*|a^*) = \frac{\lambda}{K}\,\frac{\text{Sinh}\,2\lambda L - \text{Sin }2\lambda L}{\text{Cosh}^2\,\lambda L + \text{Cos}^2\,\lambda L} \tag{3.5.25}$$

(d) Free-Free ends

$$a^* = L/2$$

$$g(a^*|a^*) = \frac{\lambda}{2K} \frac{\text{Cosh } \lambda L + \text{Cos } \lambda L + 2}{\text{Sinh } \lambda L + \text{Sin } \lambda L}$$

or $$g(a^*|a^*) = \frac{2\lambda}{K} \frac{\text{Cosh } \lambda L/2 \ \text{Cos } \lambda L/2}{\text{Sinh } \lambda L + \text{Sin } \lambda L}$$, depending upon the value of λL

Putting $g(a^*|a^*) = u_s$, bound on displacement can be obtained as

$$\int_0^L S(x) \left(\frac{\partial^2 W}{\partial x^2}\right)^2 dx > \frac{1}{u_s} W^2 \tag{3.5.26}$$

Combining Eq. (3.5.12) and (3.5.27), the bound on the displacement can be written as

$$|W(x,t)| \leq u_s^{\frac{1}{2}} \int_0^t \alpha(\tau) \, d\tau \quad \text{for } 0 \leq x \leq L \tag{3.5.27}$$

where $\alpha(\tau) = \left[\int_0^L \frac{q^2(x,t)}{m(x)} dx\right]^{\frac{1}{2}}$

If loading can be separated in the form

$$q(x,t) = \phi(t) \, f(x) \tag{3.5.28}$$

Eq. (3.5.27) can be written as

$$|W(x,t)| \leq g_s^{\frac{1}{2}} \int_0^t |\phi(\tau)| \, d\tau \quad 0 \leq x \leq L \tag{3.5.29}$$

where

$$g_s = u_s \int_0^L \frac{f^2(x)}{m(x)}\, dx$$

Thus for general load q(x,t) Eq. (3.5.27) furnishes the upper bound on displacement and for the load q(x,t) of the form given by Eq. (3.5.28) the inequality given by Eq. (3.5.29) provides an upper bound on the response of the beam resting on elastic foundation.

3.6 IMPACT LOAD.

If the q(x,t) is given by Hertz Law of impact, then

$$\left.\begin{aligned} q(x,t) &= m_2\, v_o(1+e)\, \frac{\pi}{2T_L}\, \text{Sin}\, \frac{\pi t}{T_L} && \text{for } t < T_L \\ &= 0 && \text{for } t > T_L \end{aligned}\right| \qquad (3.6.1)$$

For free-free beam on elastic foundation,

$$u_s^{\frac{1}{2}} = [\frac{\lambda}{2K}\ \frac{\text{Cosh}\ \lambda L + \text{Cos}\ \lambda L + 2}{\text{Sinh}\ \lambda L + \text{Sin}\ \lambda L}]^{1/2}$$

Then bound on displacement is given by

$$|W(x,t)| = \{\frac{\lambda}{2K}\ [\ \frac{\text{Cosh}\ \lambda L + \text{Cos}\ \lambda L + 2}{\text{Sinh}\ \lambda L + \text{Sin}\ \lambda L}]\}^{1/2} \int_0^t \alpha(\tau)\ d\tau$$

Putting the value of q(x,t) from Eq. (3.5.1)

$$\alpha(\tau) = m_2\ v_o\ (1+e)\ \frac{\pi}{2T_L}\ \text{Sin}\ \frac{\pi t}{T_L}\ \sqrt{L/\rho A}$$

$$\text{so} \int_0^t \alpha(\tau)\, d\tau = \sqrt{L/\rho A}\; \frac{m_2 v_o (1+e)\pi}{2T_L} \int_0^{T_L} \text{Sin}\; \frac{\pi t}{T_L}\, dt$$

$$|W(x,t)| = \left[\frac{\lambda}{2K}\; \frac{\text{Cosh}\;\lambda L + \text{Cos}\;\lambda L + 2}{\text{Sinh}\;\lambda L + \text{Sin}\;\lambda L}\right]^{1/2} \frac{m_2\; v_o(1+e)}{2}\; \sqrt{L/\rho A} \qquad (3.6.2)$$

For a beam with the following parameters

$\lambda = 0.080$, $\lambda L = 3.20$, $L = 40$ inch , $m_2 = 3.75$

$v_o = 9.97$ inch/sec., $e = 0.220$, $\rho = 0.29$ lb/inch^3 $A = 1.0$ inch^2

So $|W(x,t)| = 29.8 \times 10^{-3}$ inches.

This is about two times as predicted by Modal analysis and three times as predicated by Energy Method.

3.7 RESULTS AND DISCUSSION

The displacement bound within the elastic range is given by Eq. (3.2.33) and the permanent deformation is given by Eq. (3.2.34). For a half sine impulse, the displacement parameter, D and impulse parameter S, have been shown in Figs. (3.2) and (3.3) for simply supported and free-free beam on elastic foundation for different values of λL. These values are also given in Table (3.1) and (3.2). The Fig. (3.4) shows the impulse and displacement parameter for a Simply Supported beam, given by Martin [19], and Seiler, Cotter and Symonds [30]. It is observed that for beams on elastic foundation, comparatively for a smaller value of impulse parameter S, the response is in elastic range as compared to beams without foundation. For a half sine impulse, $v(x) = v_o \, \text{Sin} \, \pi x/L$, the upper bound on displacement (within elastic range) given by Eq. (3.4.5) is 12% higher than the exact value, evaluated by Modal Analysis, as in Eq. (3.4.6). For $K^o = \frac{1}{2} m_2 v_o^2$, i.e. a central impact by a mass m_2 with a velocity v_o the computed bound is 1.30 times the maximum displacement given by Modal Analysis for Simply Supported beam.

The displacement bounds for a dynamic load $q(x,t)$ using variational principles has been given in Eq. (3.5.28) and Eq. (3.5.30). These boundshave been computed for Hertz impact force given by Eq. (3.5.2). For a free-free beam, this is about two times more than that predicted by Modal Analysis and three times as predicted by Energy Method,

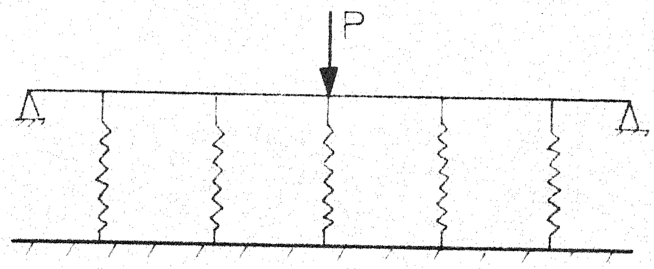
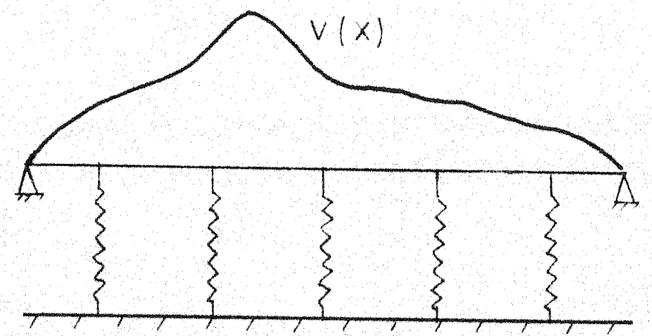


Fig.3.1(a) Statically admissible state



Initial velocity field

Fig. 3.1(b) Kinematically admissible state

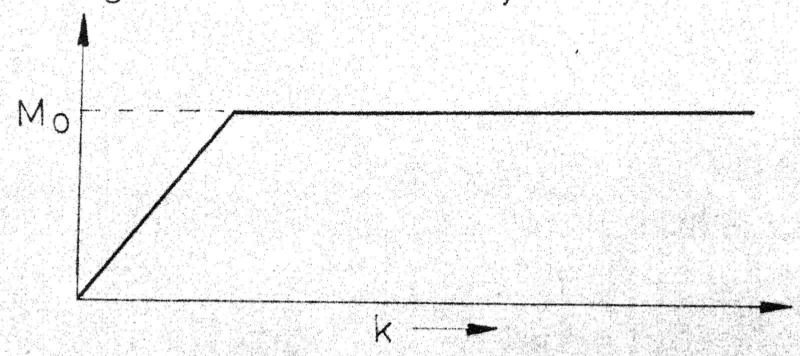


Fig.3.1(c) Moment curvature relationship

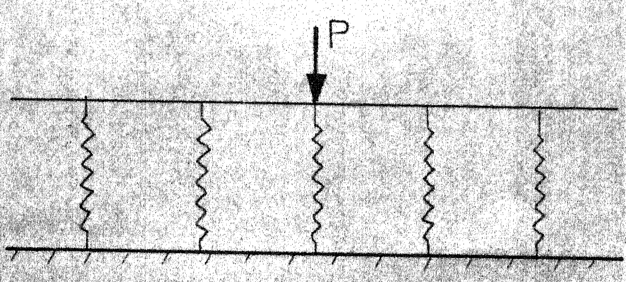


Fig. 3.1(d)

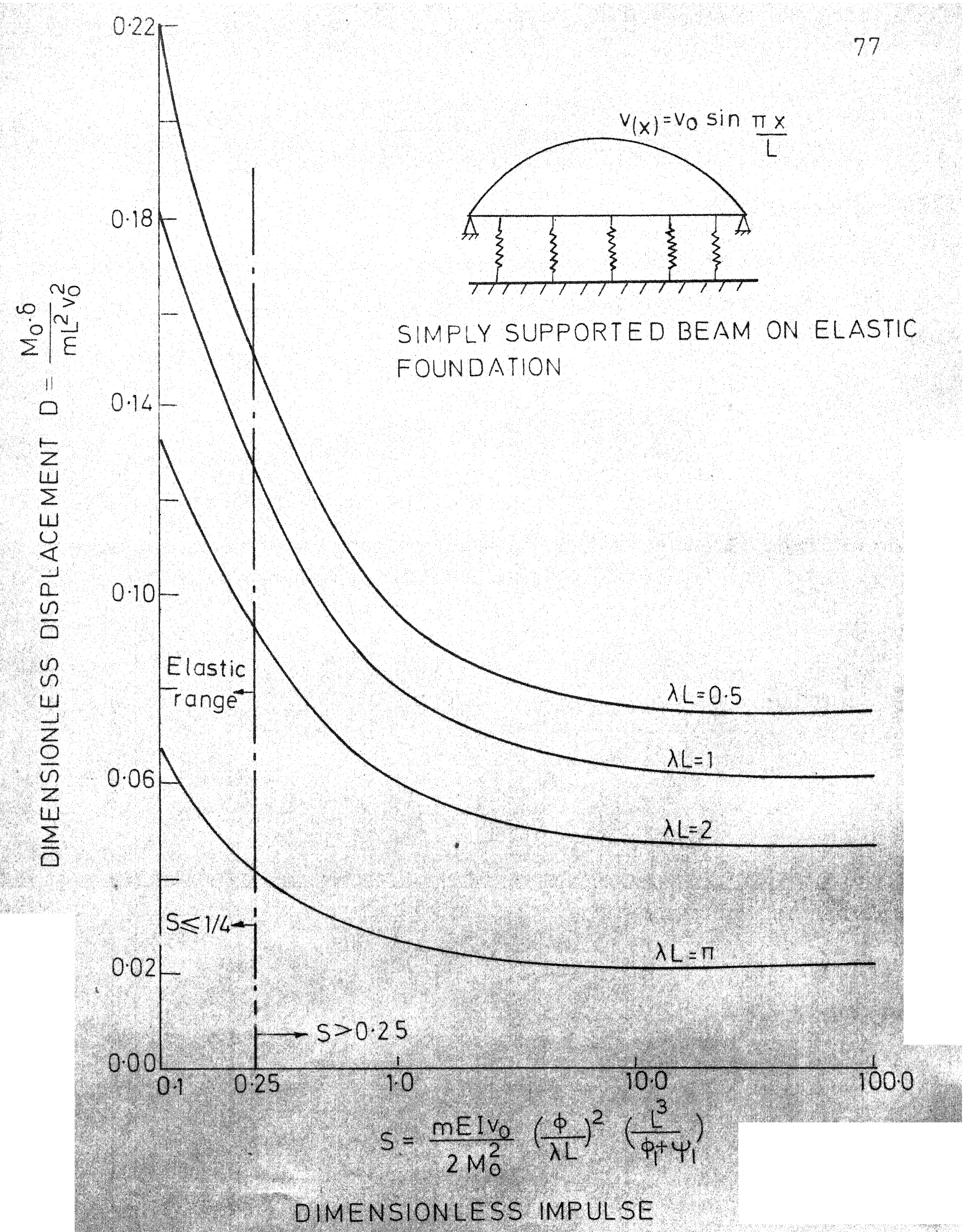
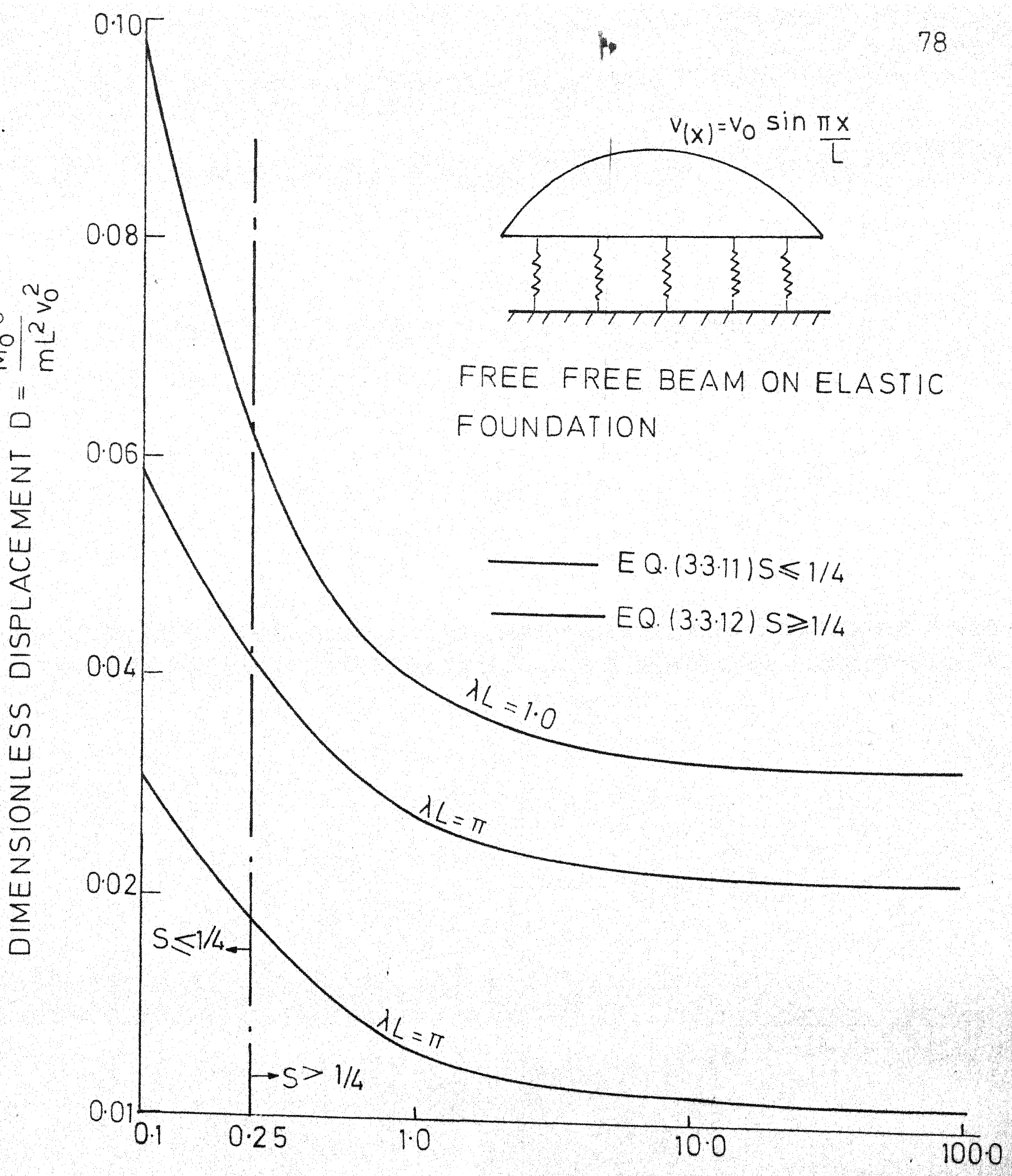


FIG. 3·2 VARIATION OF IMPULSE WITH DISPLACEMENT



$$S = \frac{mEI\,v_0^2}{2\,M_0^2}\left(\frac{\phi}{\lambda L}\right)^2 \frac{L^3}{\phi_1+\psi_1}$$

DIMENSIONLESS IMPULSE

FIG. 3.3 VARIATION OF IMPULSE WITH DISPLACEMENT

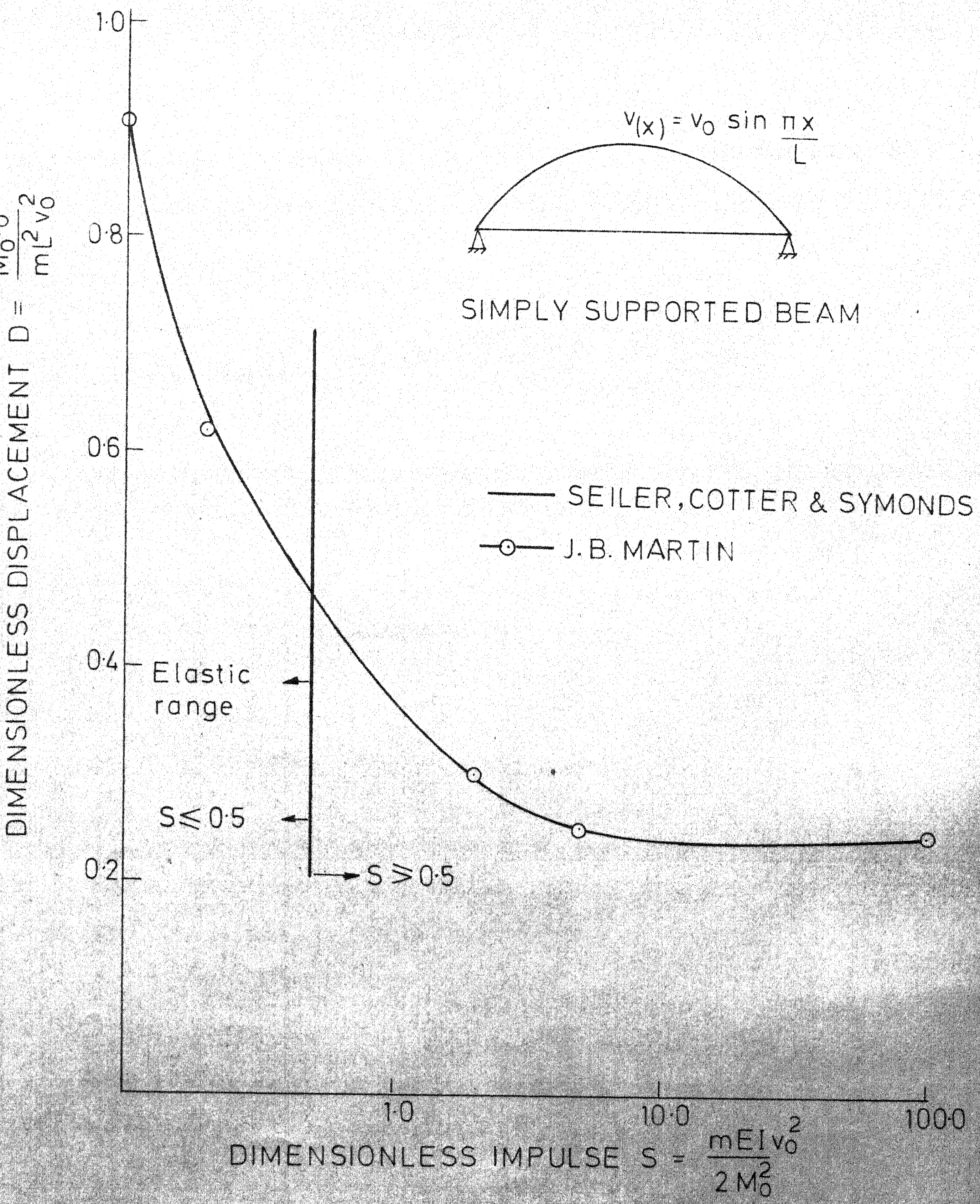


FIG. 3.4 VARIATION OF IMPULSE WITH DISPLACEMENT

CHAPTER 4

PROPAGATION OF WAVES IN BEAMS ON ELASTIC FOUNDATION AFTER TRANSVERSE IMPACT

4.1 INTRODUCTION

The study of wave motion of a beam on elastic foundation undergoing flexural vibration after impact is presented here to visualise how the flexural pulse and energy propagate in the beam. The medium can be classified as dispersive or non-dispersive depending on the propagation of waves.

4.2 BASIC EQUATIONS

If the shear deformation and rotatory inertia effects are neglected, the differential equation for the free transverse vibration of the uniform Euler-Bernoulli beam on Winkler Foundation is given by

$$\frac{\partial^4 W(x,t)}{\partial x^4} + \frac{\rho A}{EI} \frac{\partial^2 W(x,t)}{\partial t^2} + \frac{K}{EI} \cdot W(x,t) = 0 \qquad (4.2.1)$$

Assuming that the solution of Eq. (4.2.1) is a simple harmonic wave travelling with velocity v in the positive x-direction, so that its form is

$$W(x,t) = B' \operatorname{Cos} \frac{2\pi}{\theta} (x-vt) \qquad (4.2.2)$$

Where θ is the wave length and B' is a constant. Substituting Eq. (4.2.2) in (4.2.1), the velocity of propagation of Sinusoidal wave is

$$v = \frac{2\pi}{\theta} \sqrt{\frac{EI}{\rho A}} \sqrt{1 + \frac{K}{EI} \cdot \frac{\theta^4}{16\pi^4}} \tag{4.2.3}$$

Where v = phase velocity.

For K = 0, i.e. for beam without foundation

$$v = \frac{2\pi}{\theta} \sqrt{\frac{EI}{\rho A}} \tag{4.2.3a}$$

So the waves move faster in beams on elastic foundation. Therefore the velocity of propagation of Sinusoidal flexural wave is not-constant but varies with wavelength as given by Eq. (4.2.3). Usually the velocity of propagation v of a Simple Harmonic wave is called wave velocity or phase velocity.

A non-harmonic flexural pulse [23,24] may be regarded as consisting of a super position of harmonic waves of different wavelengths. Each of these waves have a different phase velocity, so it follows that a fluexural wave of arbitrary shape cannot propagate along the beam without dispersion, which results in a change in the shape of the pulse. A medium exhibiting a wave velocity depending upon the wavelength is called a dispersive medium. Hence, this uniform beam on elastic foundation is a dispersive medium for propagation of flexural pulse. The only flexural wave profile propagating in a uniform beam without alteringits shape is the simple harmonic wave. A pulse consisting of a group of harmonic waves is called a wave packet [23]. And the velocity with which such a group of waves is propagated is called Group Velocity, denoted by U. The group velocity is the velocity with which the energy is propagated.

Considering a wave packet consisting of two simple harmonic waves of equal amplitudes, of wavelengths θ and $\theta + \Delta\theta$ and of wave velocities v and $v + \Delta v$, respectively, the time interval necessary for crest 'e' to catch up with crest E, as shown in Figure (4.1) is denoted by τ. The new common positions are denoted by e' and E', respectively. Wave I travels with velocity v, wave II with velocity v + dv, and the packet with group velocity U [34]. From the Figure (4.1),

$$\left.\begin{aligned} \Delta\theta + (v + \Delta v)\,\tau &= v\tau \\ \theta + v\tau &= U\tau \end{aligned}\right| \qquad (4.2.4)$$

Combining Eq. (4.2.4) and letting the wavelength difference become very small, the group velocity expression can be obtained as

$$U = v - \theta \frac{dv}{d\theta} \qquad (4.2.5)$$

The Simple harmonic wave described by Cos $\frac{2\pi}{\theta}(x-vt)$ is called the carrier wave and the envelope shown in Fig. (4.1) represented by 2B' Cos $(\frac{2\pi}{\theta})(x-Ut)$, is regarded as moving with group velocity U.

Again considering the propagation of flexural waves and substituting Eq. (4.2.2) in Eq. (4.2.5)

$$U = v - \theta \frac{dv}{d\theta}$$

$$= v \left\{1 + \frac{(\frac{2\pi}{\theta})^2 \frac{EI}{\rho A} - \frac{K}{\rho A}(\frac{\theta}{2\pi})^2}{(\frac{2\pi}{\theta})^2 \frac{EI}{\rho A} + \frac{K}{\rho A}(\frac{\theta}{2\pi})^2}\right\}$$

For K = 0, U = v (1 + 1) = 2v. $\qquad$ (4.2.6)

But for beams on elastic foundation obviously

$$U < 2v \tag{4.2.7}$$

As the wavelength θ approaches zero, both the wave and group velocities approach infinity. The group velocity is the velocity with which energy propagates. So it can be concluded that for a pulse consisting primarily of infinitely short waves, the energy is propagated with infinite velocity, which does not seem physically possible [34].

The reason for this puzzling result can possibly be explained by the fact that for very short waves one cannot neglect rotatory inertia. When wavelength is of the same order of magnitude as the depth of the beam, the rotatory inertia effect is of the same order of magnitude as the translational inertia effects [34].

So neglecting shear deformation but considering the rotatory inertia effects in the formation, the differential equation of motion for a uniform beam on elastic foundation becomes

$$\frac{\partial^4 W}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 W}{\partial t^2} - \frac{K_g^2}{EI} \cdot \rho A \frac{\partial^2 W}{\partial x^2\, \partial t^2} + \frac{K}{EI} W = 0 \tag{4.2.8}$$

Where K_g = radius of gyration.

Assuming a solution in the form given by Eq. (4.2.2) the wave velocity can be obtained as

$$v = \sqrt{E/\rho}\ \{1 + \frac{\theta^2}{4\pi^2 K_g}\}^{-\frac{1}{2}} (1 + \frac{K}{EI}\frac{\theta^4}{16\pi^4}) \tag{4.2.9}$$

Where $\sqrt{E/\rho}$ is the wave propagational velocity for the longitudinal impacts

For K = 0, i.e. for beams without foundation

$$v = \sqrt{E/\rho}\ \{1 + \frac{\theta^2}{4\pi^2\ K_g}\}^{-1/2} \tag{4.2.10}$$

Similarly the group velocity becomes

$$U = \sqrt{E/\rho}\ \{1 + \frac{\theta^2}{4\pi^2\ K_g}\}^{-1/2} \times \{1 + \frac{K}{EI}\frac{\theta^4}{16\pi^4}\} \times \left[\frac{1 + 4(1 + \frac{\theta^2}{4\pi^2\ K_g})^{3/2}}{(1 + \frac{EI}{K}\ \frac{16\pi^4}{\theta^4})} + \theta^2(\frac{1}{\theta^2 + 4\pi^2\ K_g^2})\right] \tag{4.2.11}$$

For K = 0,

$$U = \sqrt{E/\rho}\ \ \{1 + \frac{\theta^2}{4\pi^2\ K_g}\}^{-1/2}\ (1\ + \frac{1}{1+4\pi^2\ K_g^2/\theta^2})\} \tag{4.2.12}$$

Hence in Eq. (4.2.11) and (4.2.12), it can be seen that when $\theta \to 0$, group and wave velocity tend to the constant value $\sqrt{E/\rho}$.

4.3 RESULTS AND DISCUSSION

The wave and group velocity for beams on elastic foundation and for beam without foundation has been plotted in Fig. (4.2). For beams on elastic foundation the **flexural** wave velocity is also less than the group velocity. The group velocity with which energy propagates for beams without foundation is twice the **flexural** wave velocity, whereas for beams on elastic foundation, it is less than twice the **flexural** wave velocity. However these differences are only pronounced for higher wave lengths. The **flexural** wave moves faster

in a beam on elastic foundation than in a beam without elastic foundation, whereas energy moves at a slower rate along a beam on elastic foundation than along a beam without foundation, which is shown in Fig. (4.2). The beam on elastic foundation is a dispersive medium for wave propagation, as wave velocity is dependent on wave length, as in Eq. (4.2.3).

TABLE 4.1

$$\frac{K}{EI} = 0.10 , \quad EI/A = 1.0$$

WAVE LENGTH	WAVE VELOCITY FOR BEAM	WAVE VELOCITY FOR BEAM ON ELASTIC FOUNDATION	GROUP VELOCITY FOR BEAM	GROUP VELOCITY FOR BEAM ON ELASTIC FOUNDATION
1.0	6.280	6.281	12.56	12.5000
2.0	3.140	3.142	6.28	6.2500
4.0	1.570	1.573	3.14	3.1000
5.0	1.255	1.256	2.510	2.5000
10.0	0.628	0.808	1.256	0.7720
20.0	0.314	0.768	0.628	0.0568

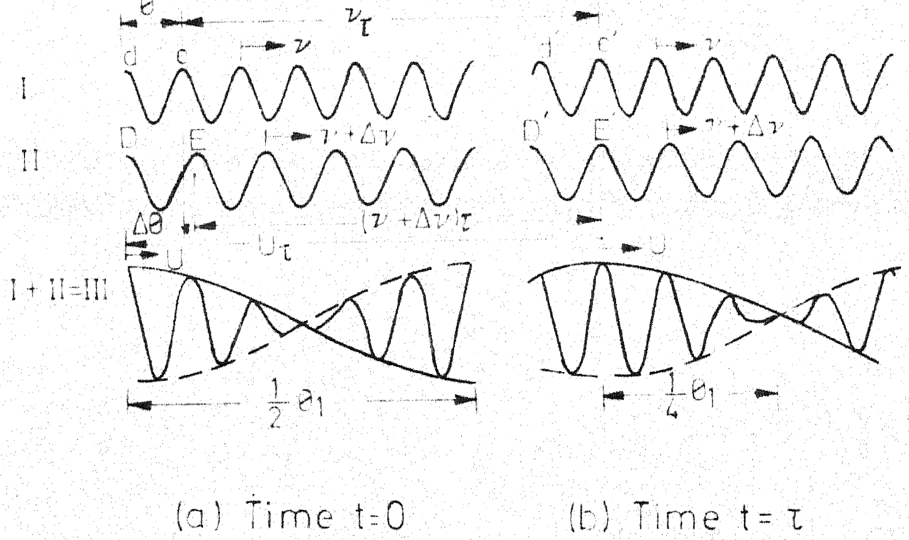


FIG. 4.1 FLEXURAL PULSE AND GROUP WAVES DISPERSIVE MEDIUM

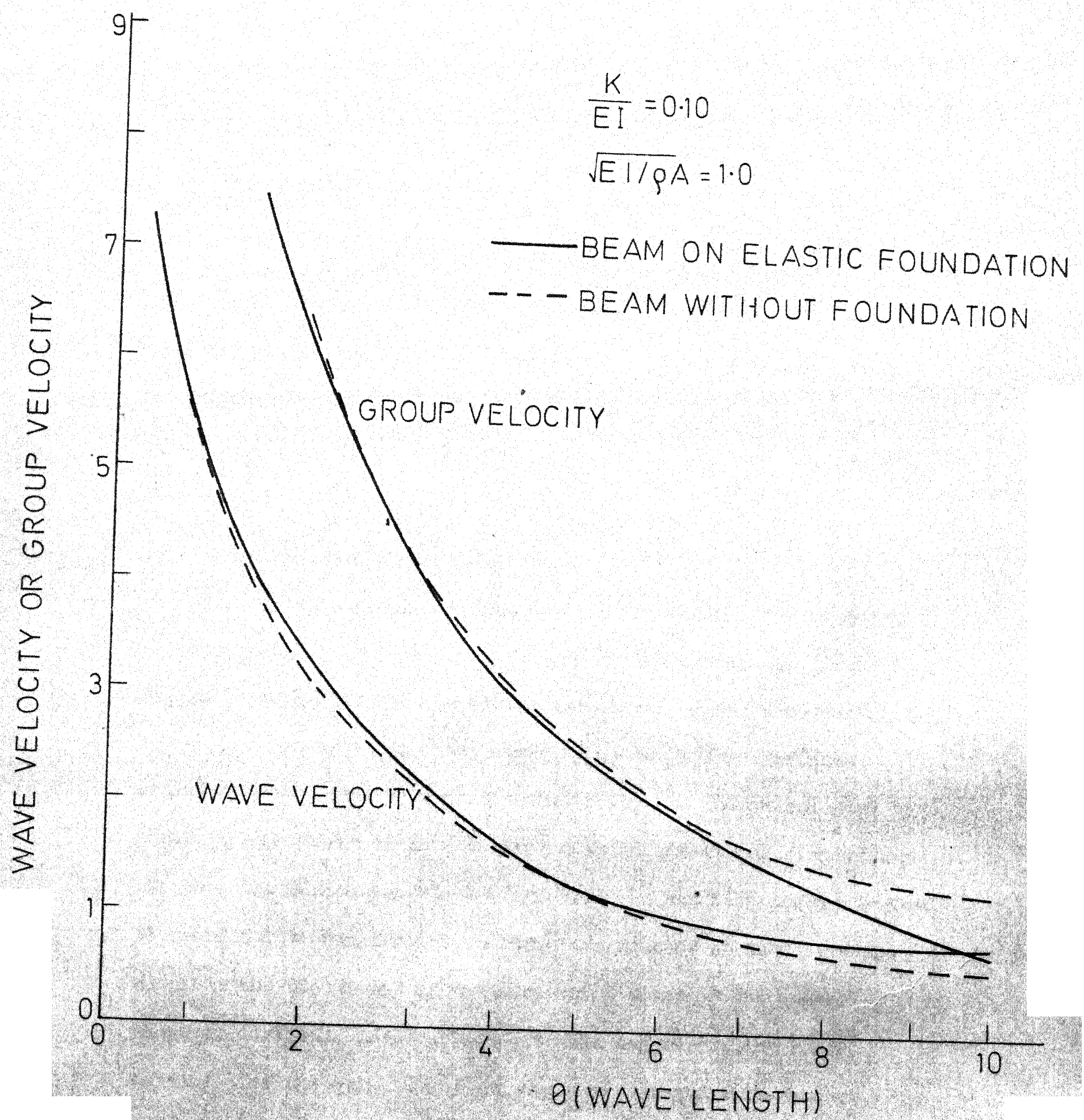


FIG. 4·2 VARIATION OF WAVE VELOCITY WITH WAVE LENGTH

CHAPTER 5

EXPERIMENTAL SET-UP AND MEASUREMENTS

5.1 INTRODUCTION :

To study the response of beam on elastic foundation, experimentally, a suitable set-up is needed to do so. The set-up in the Dynamics and vibration Laboratory of Mechanical Engineering Department, I.I.T. Kanpur is used for the experimental studies reported here. The details of the set up are shown in fig.(5.1,5.2) and in plate (2, 3,4,5,9).

5.2 BRIEF DESCRIPTION OF SET-UP :

An experiment has been designed to verify the theoretical predictions[12,13,32,35] The experimental arrangement is given in figure (5.1, 5.2). Details of the set-up are shown in plates (2, 3, 4, 5,9). The hammer is supported by means of low stiffness stainless steel tube to minimize the energy absorption and also to ensure the movement of the striking mass in a vertical plane. At the top, the suspension block (fig. 5.2) has been precisely made to avoid frictional forces. Sliding arrangement at the top makes possible the adjustment of the suspension point of the hammer whenever desired. Scale attached to the top supporting block measures (of the order 0.5°) the angle of swing for the hammer.

Neglecting the friction losses and assuming pendulum to be a simple one, expression for striking velocity v_o is obtained by equating the maximum potential and kinetic energies of the hammer; and can be written as

$$v_o = 2gR\,(1 - \cos\,\theta)^{\frac{1}{2}} \qquad (5.2.1)$$

where R = Length of Pendulum.

5.3 EXPERIMENTAL MODEL:

Beam of mild steel and of uniform cross-section was put on ten almost identical linear springs at a spacing of four inches. The details of the characteristics of the beams and springs used for the experimental study are given in Table 5.1. With a help of washer, nuts and bolts beam and springs were connected such that different beams can be attached to the springs. These springs were welded to a fairly heavy channel making the base rigid. The ends of the beam were kept free-free. Hence a uniform free-free Euler-Bernoulii beam resting on Winkler model of foundation was simulated in the experimental set-up as shown in Plate 1.

5.4 INSTRUMENTATION :

In studying experimentally the response of the beam subjected to transverse impact, various parameters such as bending strain, deflection, acceleration, velocity of the beam have been measured. The very short duration of the impact (of the order 100 micro seconds) needs effective design of the electronic circuitry

as well as the proper selection of the transducers for measuring various parameters involved. The details are discussed in the following sections.

5.5 SPRING CONSTANT:

All the ten springs which were used in the model were tested on a uni-axial compression testing machine. It was found that the Load-Deflection relationship was fairly linear during loading, unloading and reloading. After fabricating the model, it was put on platform of a tri-axial testing machine (AIMIL). A proving ring was used to measure the load at the centre of the beam and deflection, at centre (i.e. below the load pt), was measured with a dial gage (L.C = 0.01 mm) . The deflection at centre under the load P, for a free-free beam on elastic foundation is given by,

$$W = \frac{P\lambda}{2K} \cdot \frac{\text{Cosh}\,\lambda L + \text{Cos}\,\lambda L + 2}{\text{Sinh}\,\lambda L + \text{Sin}\,\lambda L} \qquad (5.5.1)$$

where $\lambda^4 = \sqrt[4]{\frac{K}{4EI}}$

or $$W = \frac{PL^3}{8EI(\lambda L)^3} \quad \frac{\text{Cosh}\,\lambda L + \text{Cos}\,\lambda L + 2}{\text{Sinh}\,\lambda L + \text{Sin}\,\lambda L} \qquad (5.5.2)$$

The following observations were made

P in lbs	-	W in mm
50	-	0.56
100	-	1.1

From Eq.(5.5.2), by trial and error method

$$\lambda L = 3.2$$

So $$\lambda = 0.08 \text{ inch}^{-1}$$

$$K = 99.8 \text{ lb/in}^2$$

And width of beam = 1.98 inches.

Thus this model represented a foundation having a modulus - 50.0 lb/in^3.

5.6 STRAIN MEASUREMENTS :

Strain gages of bounded wire type have been employed for measurement of strain. At five places strain gages have been installed. At point of impact, two strain gages on the back side of the beam were installed and two dummy gages were installed, on similar material as that of beam . At other places one strain gage on each side of the beam were cemented.

The leads from the strain gages were connected to Type 3C66 carrier Amplifier (Bridge Balance) plug-in units. The plug-in unit is designed to be used with 564-oscilloscopes. This unit is self contained and requires no external equipment other than the transducer. For strain gage at the middle, four External Arm Connection was made as in fig.(5.3). For strain gages at the other places, two External Arm Connection was made as shown in fig.(5.3).

The type 3C66 plug-in unit was inserted into the proper plug-in compartment of the oscilloscope. Strain gages were connected

to the Type 3C66. The Unit Input Bridge Circuit was balanced and with the strain gages in an unstressed condition, the bridge can be checked for proper balance by placing the FUNCTION switch in the BALANCE position. If the bridge is correctly balanced, no signal appears on the Cathode ray tube (CRT).

5.7 STRAIN-GAGE CALIBRATION:

The Type 3C66 plug-in unit can be calibrated with the standard 150K calibration resistor supplied with the unit. This resistor is stable and accurate with good electrical properties at 25 KC . Operating the Calibrate Switch on the type 3C66 unit from front panel which connects the calibration resistor in parallel with the bridge arms and EXT ARMS Connector. This reduces the resistance of the bridge arm, and is equivalent to subjecting the strain gage to a negative (compressive)strain And the equivalent strain is given by

$$\text{Equivalent strain} = \frac{R_{SG} \times 10^6}{(G.F.)(R_{SG} + R_{CAL})} \qquad (5.7.1)$$

where G.F. = Gage Factor of strain gage

R_{SG} = Strain Gage Resistance

R_{CAL} = Calibration Resistor Value

For G.F. = 2.05, R_{SG} = 113.15 ohms, R_{CAL} = 150K

Equivalent Strain = -370.

The GAIN Control is adjusted to make the STRAIN/DIV switch direct reading for strain measurements.

In equation (5.7.1), the strain simulated by the 150K calibration resistance for specific non-standard condition is -370 strain. Using this value, the type 3C66 plug in unit was calibrated at 100 strain/div sensitivity.

$$\text{Division of Trace Deflection} = \frac{-370}{100} = -3.7 \text{ divisions.}$$

The actual calibration was performed by

1. Setting the STRAIN/DIV switch to 100.
2. Trace was positioned to top of CRT.
3. Operating the 'Calibrate' switch, the trace is deflected downwards.
4. GAIN Control was adjusted for a deflection of 3.70 major gratitude divisions.

Thus the STRAIN/DIV was calibrated for direct reading at all settings.

5.8 DEFLECTION MEASUREMENTS:

The deflection at various points were measured with the help of Vibration Pick up, (of piezo-electric ceramic accelerometer-type), Vibration meter and storage oscilloscope as shown in plate . The pick-up placement was accomplished with 1560-P35 Permanent-Magnet clamp. This is a disk-shaped magnet with a

captive screw which was threaded into the tapped holes in the pick-up. With the FUNCTION switch at DISP, the vibration meter performs a double integration on the acceleration response of the pick-up. Further the deflection- time trace (plate 6) was obtained on the storage oscilloscope.

The velocity and acceleration response of the beam was obtained with the FUNCTION switch of oscilloscope at VEL and ACCEL respectively.

The calibration of the vertical scale of the oscilloscope was done with the help of an Electro-dynamic shaker and a Power Amplifier as shown in plate .

5.9 SPECIFICATIONS OF INSTRUMENTS USED :

1. Oscilloscope

a. 564 Storage Oscilloscope

This is a special purpose oscilloscope designed to store Cathode-ray tube, and displays for viewing or photographing up to an hour after application of the input signal. This can be operated as a conventional oscilloscope also. There are two separate storage screens of the Cathode-ray tube. Either the upper or lower storage screen in storage mode can be used while the other can be operated in non-storage screen.

The Plug-in units used are

1. Type 3A3 Amplifier Unit.
2. Type 3B4 Time Base Plug in Unit.

3. Type 3C66 Bridge Balance.
TEKTRONIX INC. BEAVERTON, OREGON (U.S.A.)

2. Vibration Pick-UP

This is an acceleration type piezo-electric pick-up used to measure amplitudes with the help of vibration meter. This can be held in position on any vibrating body to obtain the amplitudes, or it can be fixed to the vibrating body with magnetic attachment.

GR Type - 15 1560 - P52
SERIAL NO.... 3397

Pick-up sensitivity - 67.5 mV/g.

Manufacturers: General Radio Company, Concord, Massachusetts, U.S.A.

By Shore Brothers, Inc. Evanston, Ill., U.S.A.

3. Vibration Meter

The type 1553-AK Vibration Meter is a portable, general purpose instrument for measuring mechanical vibration when used with the Type 1560-P52 Pick-up it is direct-reading in

Acceleration (inch/second^2)
Velocity (inch/second)
Displacement (milli inches)

The vibration Meter consists of a piezo electric ceramic accelerometer -type pick-up and a sensitive vacuum-tube voltmeter. It has integrating circuits to convert acceleration to velocity or displacement

signals, a differentiating circuit to convert acceleration signals to jerk signals.

Vibration Meter Serial No.
1557-0100-C

GENERAL RADIO COMPANY
WEST CONCORD, MASSACHUSETTS, USA.

4. Carrier Amplifier (Bridge Balance)

The type 3C66 Carrier Amplifier plug-in unit is designed to be used with Tektronix 560-series osscilloscopes that will accept 3-series plug in units. The unit is self contained and requires no external equipment other than the transducer. Excitation Voltage for the transducer is provided by the above unit. Suppressed-carrier amplitude modulation produced by unbalancing an A-C bridge with strain gages. A phase sensitive demodulator produces the proper direction of trace deflection.

TECKTRONIX INC. BEAVERTON, OREGON (USA).

5. Strain Gages CA-10 (Bonded Type)

Base Length	-	10 mm
Base Width	-	6 mm
Resistance	-	111.0 ± 0.5 ohms
Gage Factor	-	2.05

Manufacturer: Mahavir Optical & Scientific Works,
44, Civil Lines, Roorkee, U.P. (India).

5.10 RESULTS AND DISCUSSION

The experimental results have been shown in Figs. (2.1 to 2.10). The deflection at X = L/2, along at other points of beam have been measured. At X = L/2, L, L/5 results have been plotted against v_o and time in Fig. (2.1, 2.1a, 2.1b). The measured values are in good agreement with theoretical values. Measured values are in between the values predicted by Modal Analysis and Energy Method. But the experimental values are in better agreement with Modal Analysis than Energy Method. In Fig. (2.2), Deflection has been plotted against time. The experimentally obtained deflection-time curve is also nearly sinuosidal in character. These traces have also been shown in plate 6 as obtained on the storage oscilloscope. At other points of the beam also similar observation have been obtained, though not shown. The deflection has been also plotted as magnification factor by dividing these values by $P\lambda/2K$.

The experimental value of strain at X = L/2, and X = L against v_o has been plotted in Fig. (2.3, 2.5). These values also linearly increase with v_o. But experimental values are higher than the analytically predicted values, at X = L/2, X = L/5. At X = L/2, the experimental values are about 1.4 times the value predicted by Modal Analysis and two times the value predicted by Energy Method. However at X = L, Fig. (2.5), the measured value of strain was 0.70 times the value predicted by analytical method and two times the value predicted by Energy Method. At X = L/5 also the measured values are higher than the theoretical value. This difference

may be due to hole in the experimental beam, made for connection with the spring which are points of stress discontinuity. The strain-time curve has been plotted in Fig. (2.4) and Fig. (2.6) for $X = L/2$ and for $X = L$ respectively and they are also shown in plate 10.

The impact force has been shown in Fig. (2.7, 2.8). The maximum experimental value of impact force is less than the maximum value predicted by Modal Analysis and more than that predicted by Energy Method. The Acceleration and velocity **time trace** has been shown in plate (8 and 7).

For measuring the displacement and acceleration, calibration of the vertical scale of oscilloscope has been done and shown in Figs. 5.5 and 5.6) respectively.

In case of strains at $x = L/2$, the difference between theoretical values and experimental values may be due to three-dimensional effects of the central impact.

TABLE 5.1

LENGTH OF ALL BEAMS = 39.98 inches

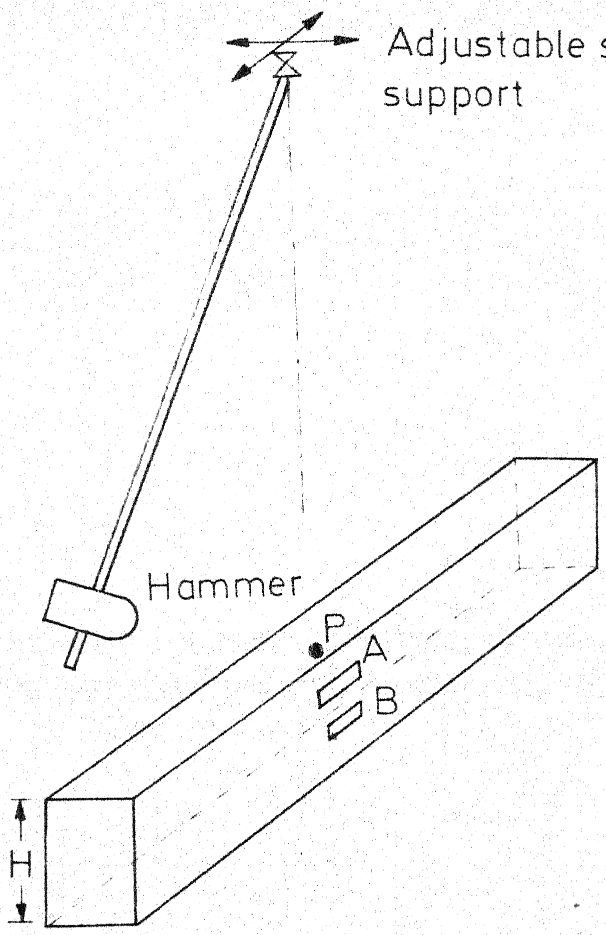
SPRING CONSTANT = 99.8 lb/in^2

No.	Beam Size		ρ	E	m_2g in	Hammer Radius
	D inch	H inch	lb/in^3	p.s.i.	lb.	inch
BEAM I	1.98	0.495	0.292	$30x10^6$	3.75	2
BEAM II	1.98	0.250	0.292	$30x10^6$	3.75	2

BEAM I

No. of Mode n	Natural Frequence = w_n		Simply Supported Beam without foundation
	Free-Free	Simply Supported	
	Beam on Elastic Foundation		
1	536	403	173
2	1154	785	695
3	1827	1607	1550
4	3538	2806	2480
5	5270	4364	4320
6	7352	6272	6200
7	9783	8531	8448
8	12563	11138	11100
9	15690	14094	14020
10	19200	17399	17300

Coefficient of restitution for free-free beam I on Elastic Foundation = e = 0.312



A, B - Strain gages connected to the oscilloscope

P - Point of impact

FIG. 5.1 LINE SKETCH FOR EXPERIMENTAL ARRANGEMENT

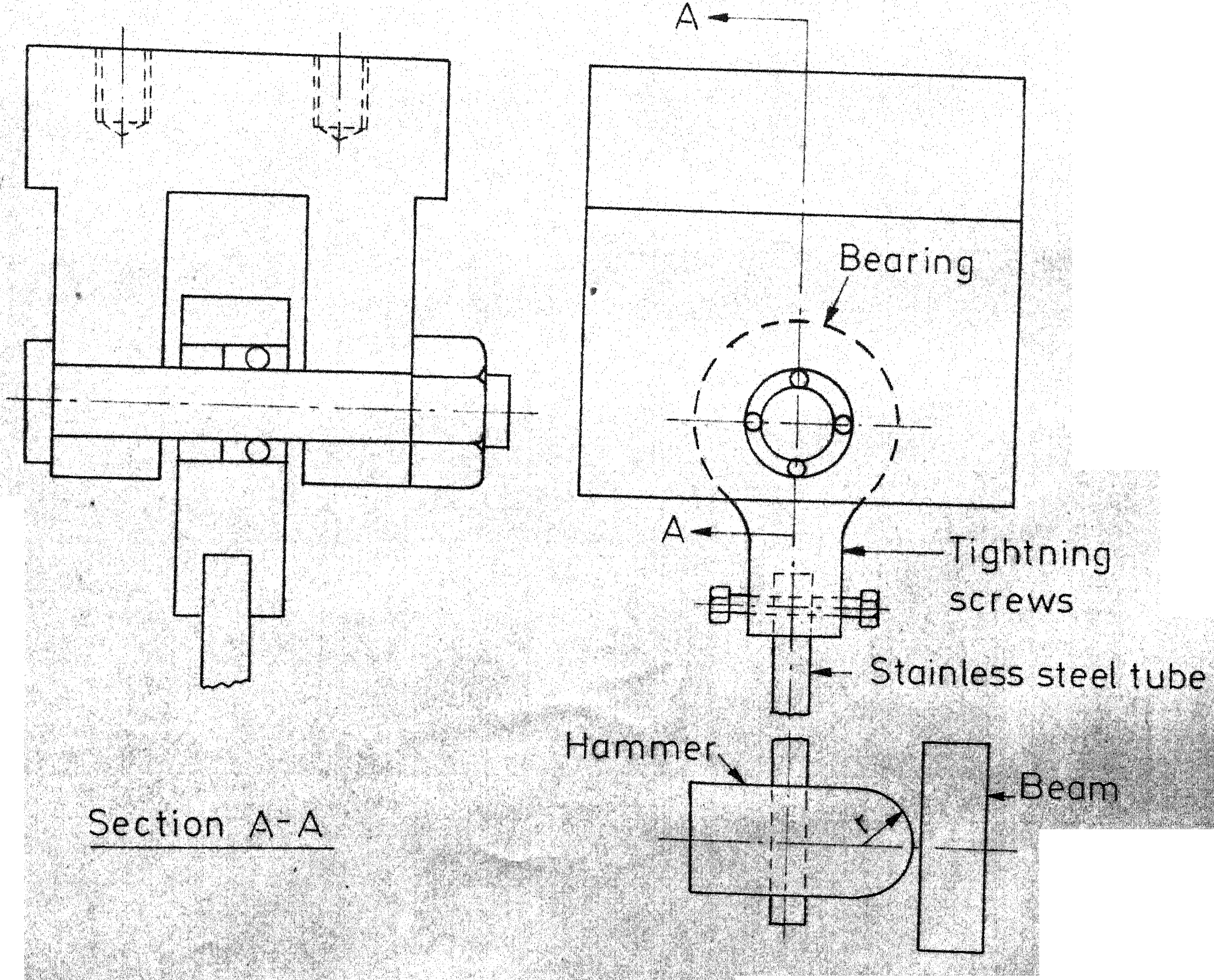
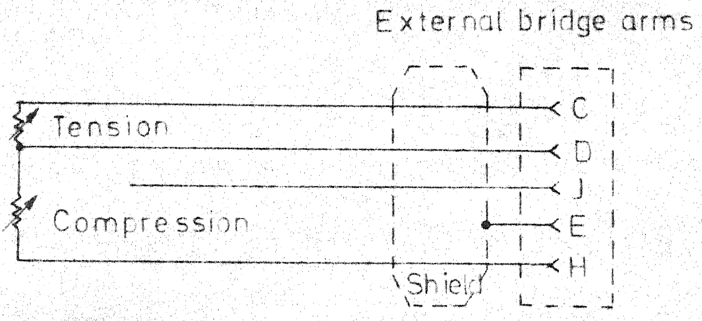
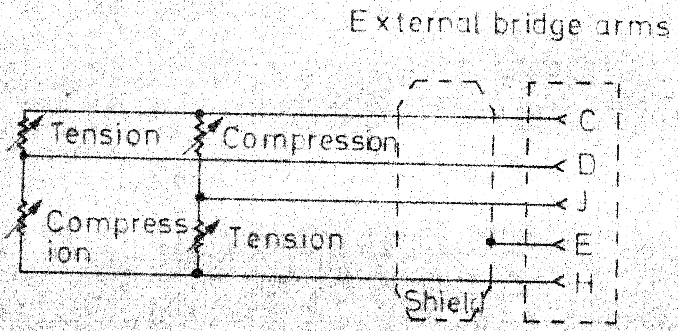


FIG. 5·2 SUSPENSION BLOCK



Two Ext. Arms



Four Ext. Arms

FIG.5.3 NORMAL CIRCUIT FOR RESISTIVE TRANSDUCERS (STRAIN GAGES)

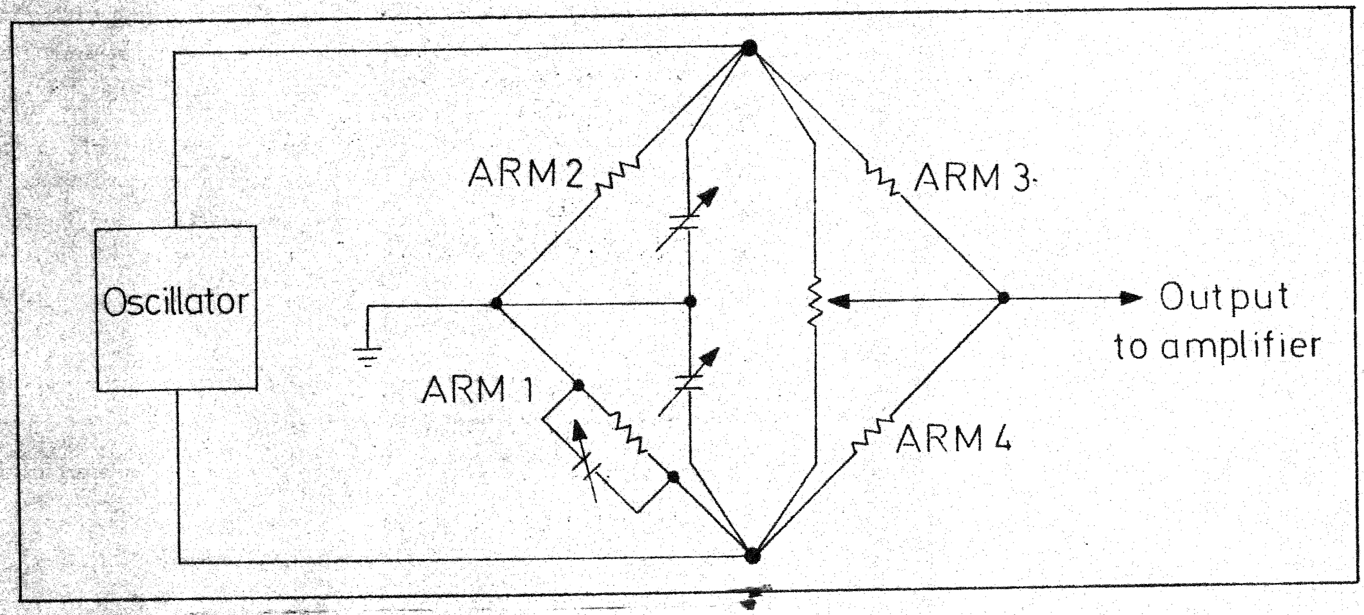
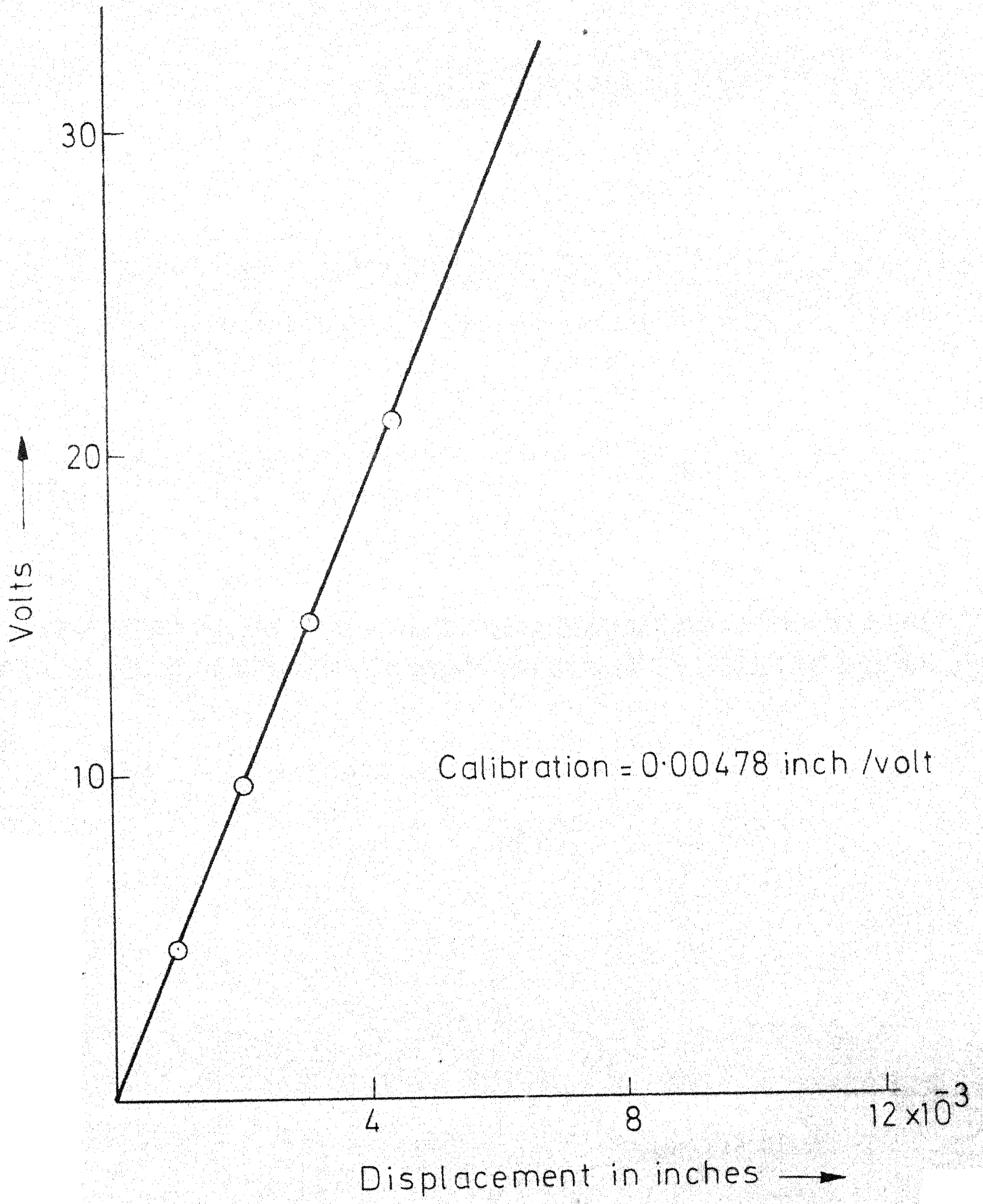


FIG. 5.4 SIMPLIFIED BRIDGE SCHEMATIC SHOWING BRIDGE BALANCE CIRCUIT



FIG·5·5 CALIBRATION OF OSCILLOSCOPE
(Displacement measurement)

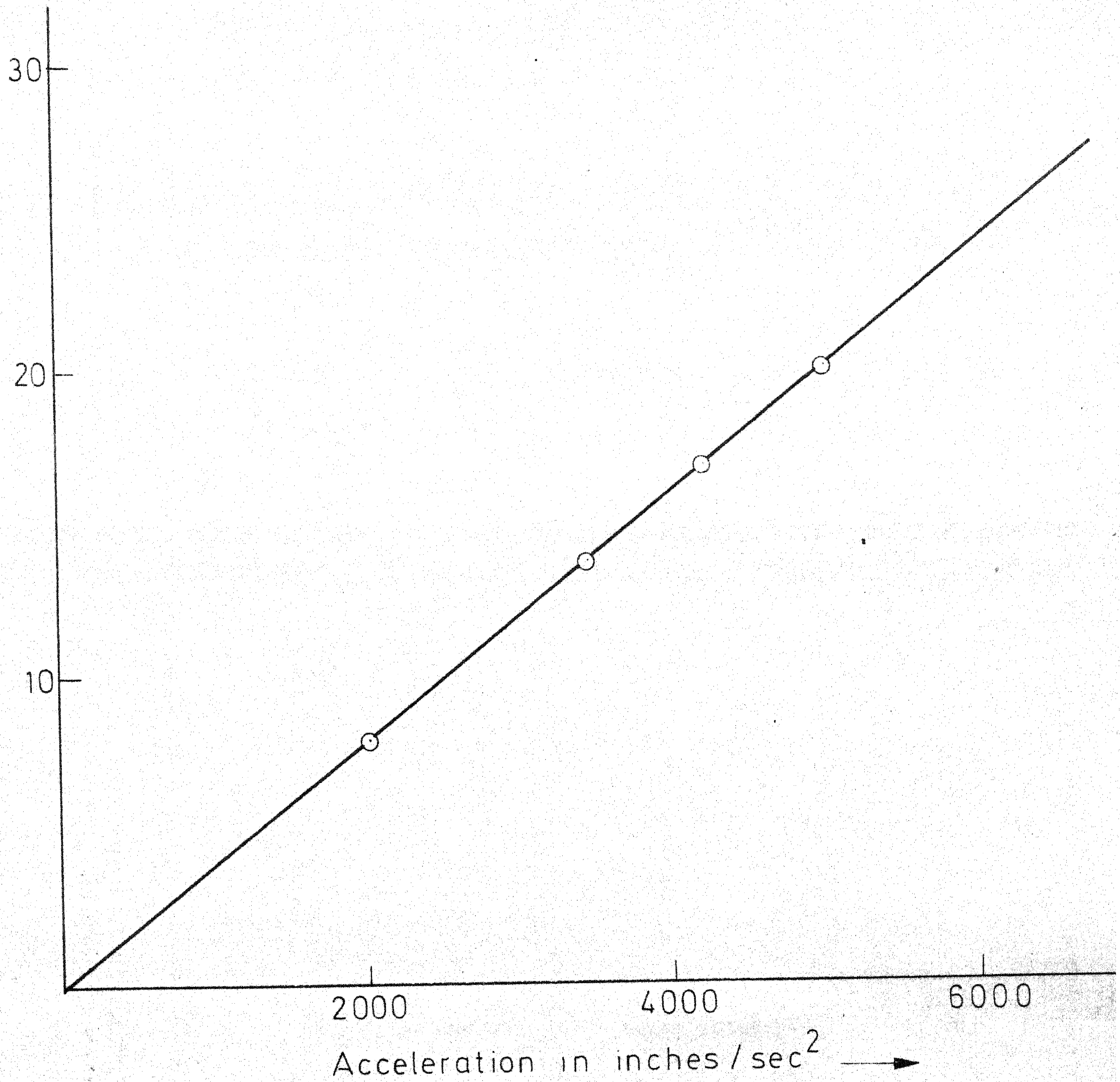


FIG. 5.6 CALIBRATION OF OSCILLOSCOPE (Acceleration measurements)

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 GENERAL CONCLUSIONS

The response of beam on elastic foundation has been analyzed by Modal Analysis and Energy Method. The entire accuracy of method using Modal Analysis depends upon the assumption of initial velocity distribution. The distribution assumed in this present analysis has not been truly represented in the experimental model, owing to the small length of the model beam. However under these limitations, they are in good agreement. Impact load on elastic foundation is essentially a problem of dynamic plasticity , and permanent deformations are of considerable importance. Bounds can be computed for such deformation using Energy Principles.

Displacement Bounds within the elastic limit can be computed using Energy theorem and also by suitable inequalities and variational technique. The study of wave propagation along the beam on elastic foundation reveals the validity of assumptions usually made in Modal Analysis or Energy Method.

The following conclusions have been made, based on the analytical and experimental results, obtained in the present investigation.

(1) The Modal Analysis and Energy Method can be used to predict the response of beams on elastic foundation, subjected to

Impact loads. According to present study, Modal Analysis is preferable to Energy Method.

(2) For free-free beam on elastic foundation, the rigid body translational mode and rotational modes modify the beam response. In the present study the deflection at the centre was more than at the free-free ends, however it is more a function of length of beam and it is also dependant rigid body modes and number of modes excited by the impact load.

(3) The effect of foundation modulus is to increase the natural frequency of thebeam on elastic foundation. However this effect diminishes as number of modes increases.

(4) For free-free beam, the first mode is predominating, and the deflection as well as strain decreases rapidly as the number of mode shape increases. However for simply supported beam, the first mode gives the major part of deflection, but for strains higher modes contribute more than the first mode.

(5) The external damping greatly reduces the deflection as well as strain. For $\xi < 1.0$ the motion is oscillatory with diminishing amplitude. And for $\xi > 1.0$, the motion is no longer oscillatory being exponentially decreasing with time.

(6) The experimental values of deflection, impact force are in fairly good agreement with theoretical values predicted by Modal Analysis, and Energy Method. However in this study agreement

is more pronounced with Modal Analysis than Energy Method. In the case of strains, the agreement is not that pronounced.

(7) For a beam on elastic foundation subjected to dynamic load and impact load, where in initial velocity distribution is known, bounds on displacement can be computed using Energy Method. Within the elastic range, for a half sine curve the computed bound is only 12% higher than the exact value. These bounds are less for beams on elastic foundation than for beam without foundation. Assuming an elastic plastic moment curvature relationship, elastic contribution towards the bound is very small. So it can be safely taken as rigid-plastic, as far as the computation of these bounds outside the elastic range are concerned. The contribution from elastic range is more in beams without foundation than beams on elastic foundation.

(8) Impact load causes permanent deformations on the beams. Without necessarily knowing the deformation process, the ultimate permanent deformation can be obtained by these bounds, computed outside the elastic range. With increase in foundation modulus these bounds decrease.

(9) For a central impact by a mass m_2 with a velocity v_o, for a simply supported beam on elastic foundation, the computed bound deflection is only 1.3 times more than that by Modal Analysis.

(10) For a general class of loading $q(x,t)$, displacement bounds can be computed for a beam on elastic foundation as given in Eq. (3.5.28) and Eq. (3.5.30) by using variational principle. For impact loads as given by Hertz, these bounds are about twice the value predicted by Modal Analysis and thrice the value predicted by Energy Method.

(11) The beam on an elastic foundation is a dispersive medium for wave propagation. The flexural wave moves faster in a beam on elastic foundation than in a beam without foundation, where as energy propagates at a slower rate along a beam on elastic foundation than along a beam without foundation.

6.2 SCOPE FOR FURTHER RESEARCH

The following topics are listed for further investigations :

1) Impact loads on long beams on Elastic foundations.

2) Experimental studies taking care of Damping in the Foundation medium

3) Experimental Methods for separating rigid body motion in the case of beams on elastic Foundations with free ends.

4) Effect of shear Deformation androtatory inertia on the responses of beams on elastic foundations subjected to Impact loads.

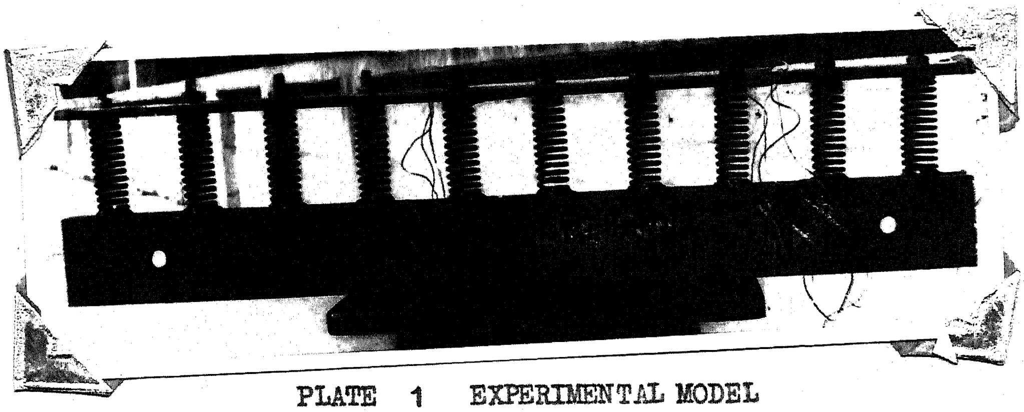


PLATE 1 EXPERIMENTAL MODEL

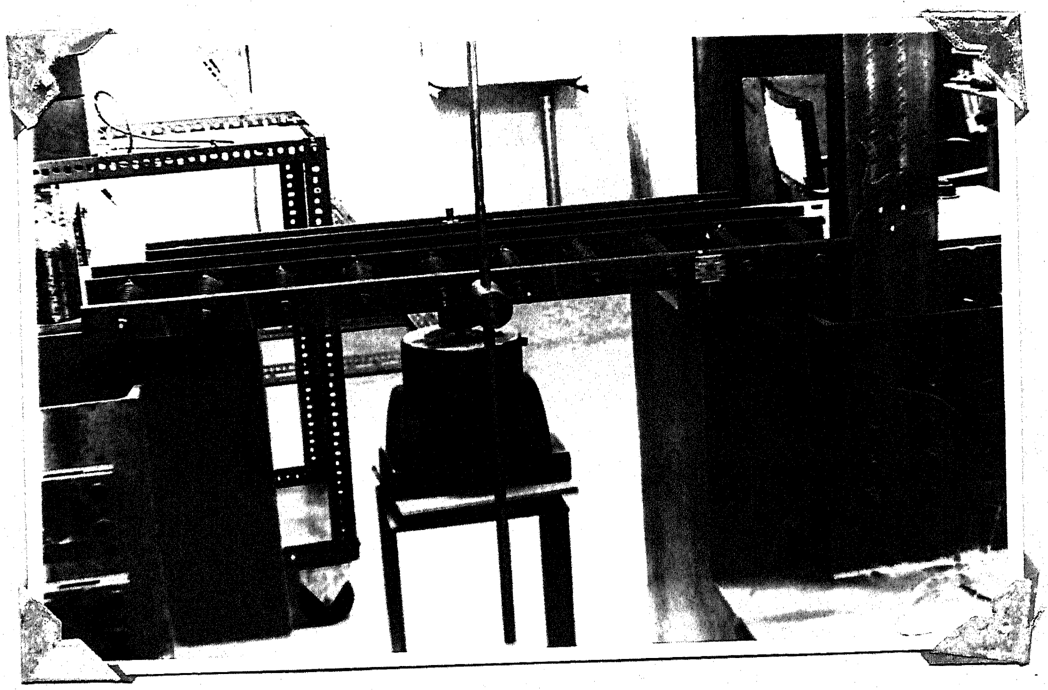


PLATE 2 LOWER PART OF SET UP

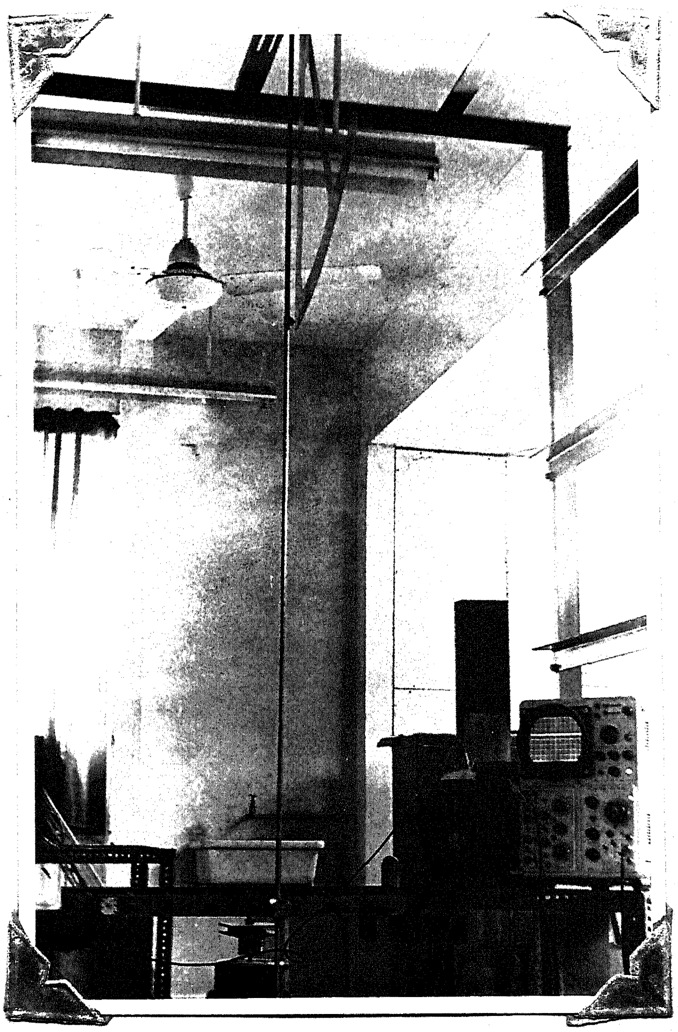


PLATE 3 UPPER PART OF SET UP

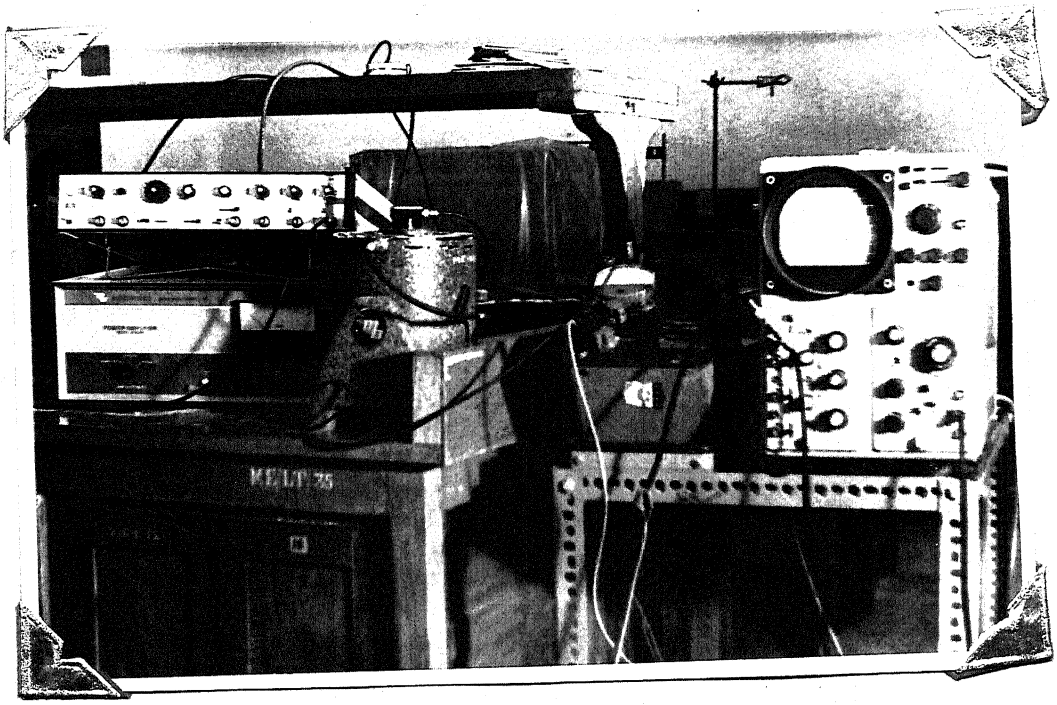


PLATE 4 SET UP FOR CALIBRATION OF ACCELEROMETER

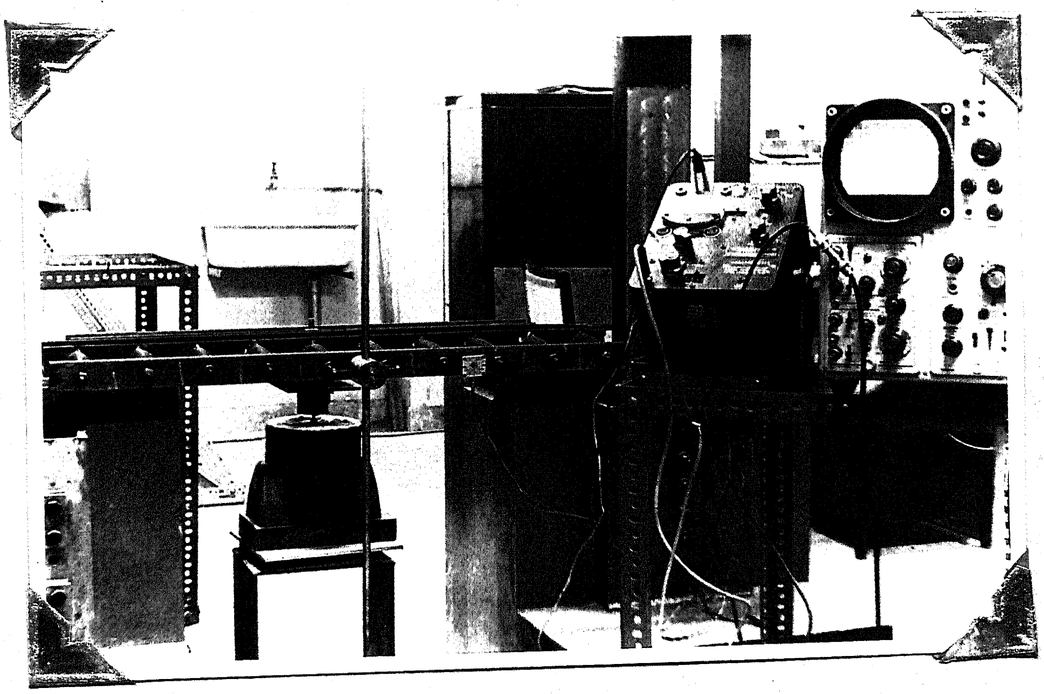
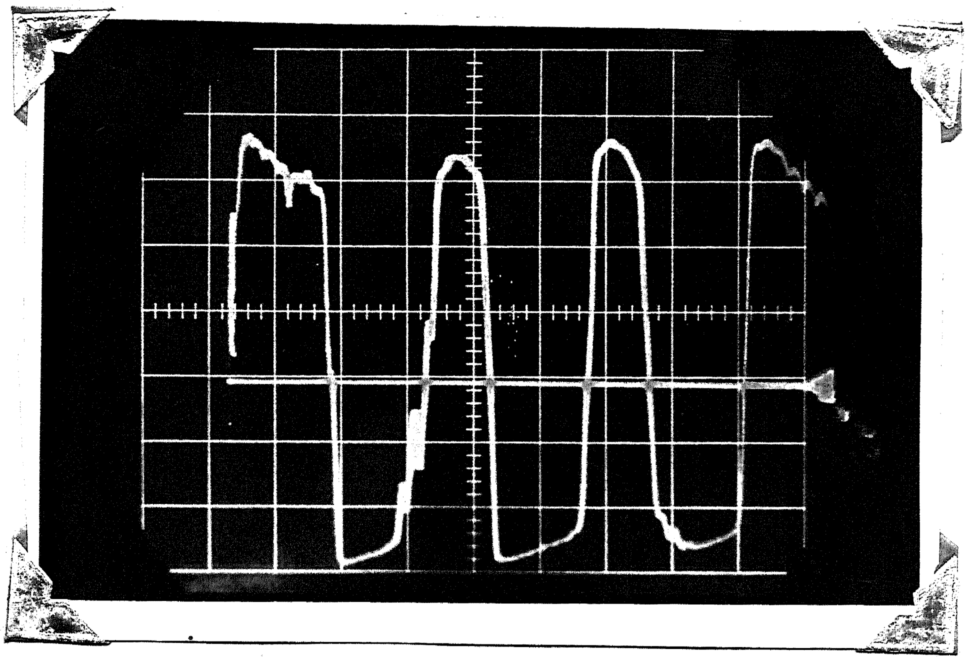
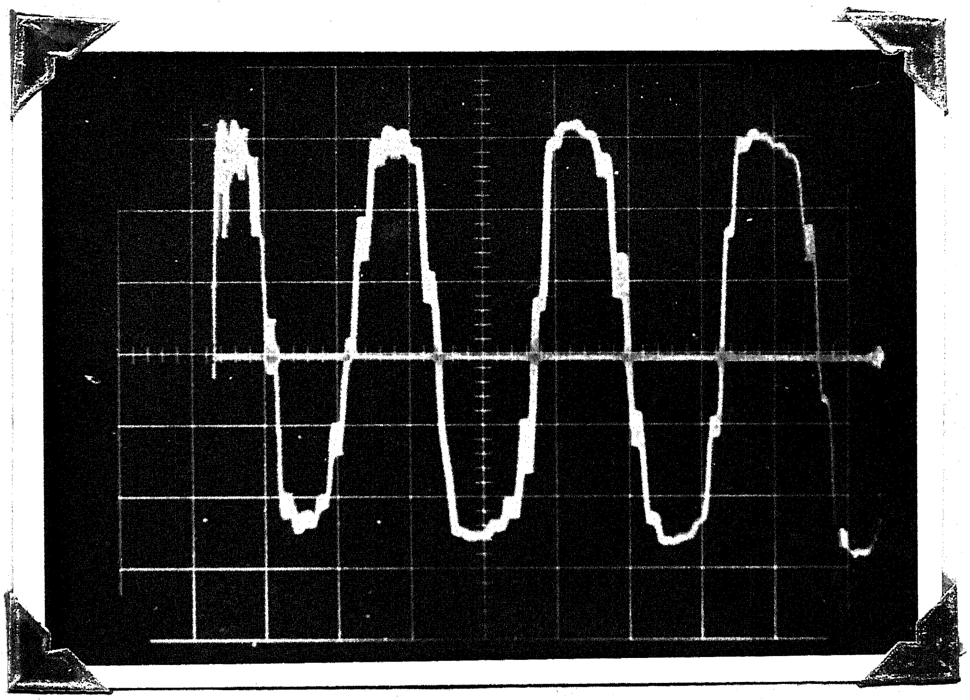


PLATE 5 SET UP FOR DEFLECTION MEASUREMENT

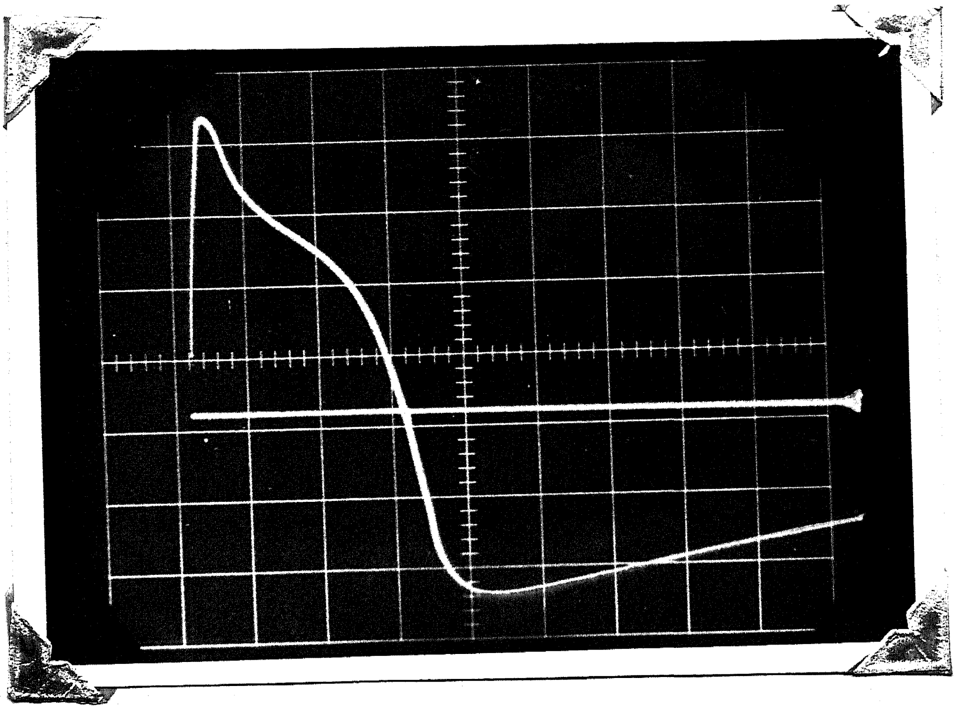


At x = L



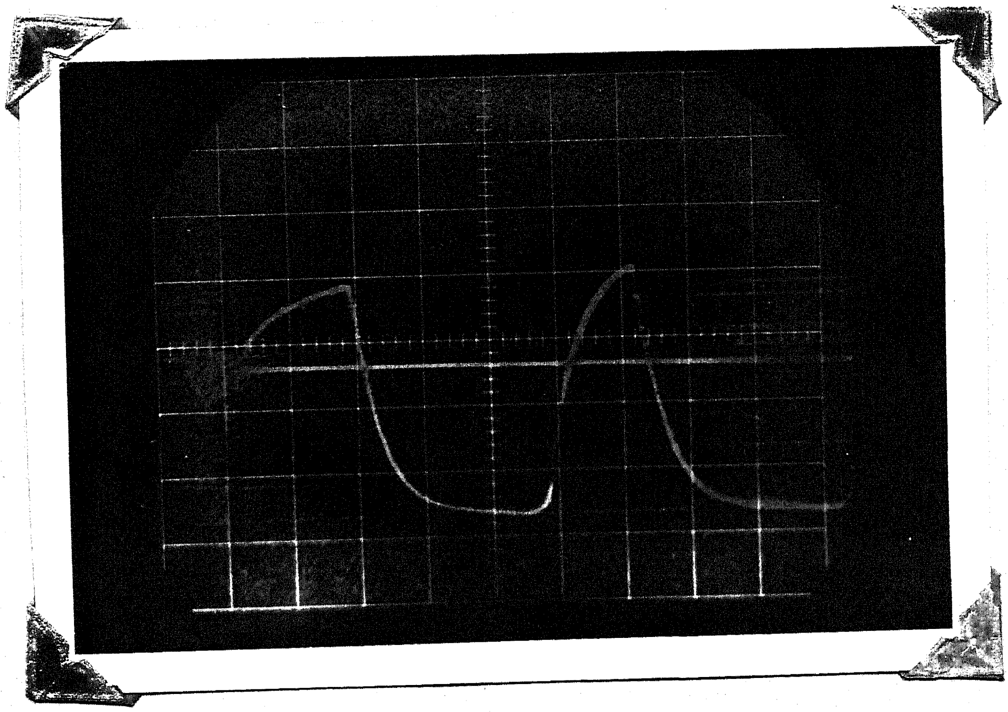
At x = L/2

PLATE 6 DEFLECTION- TIME TRACE



AT X = L/2

PLATE 7 VELOCITY- TIME TRACE



AT X = L/2

PLATE 8 ACCELERATION - TIME TRACE

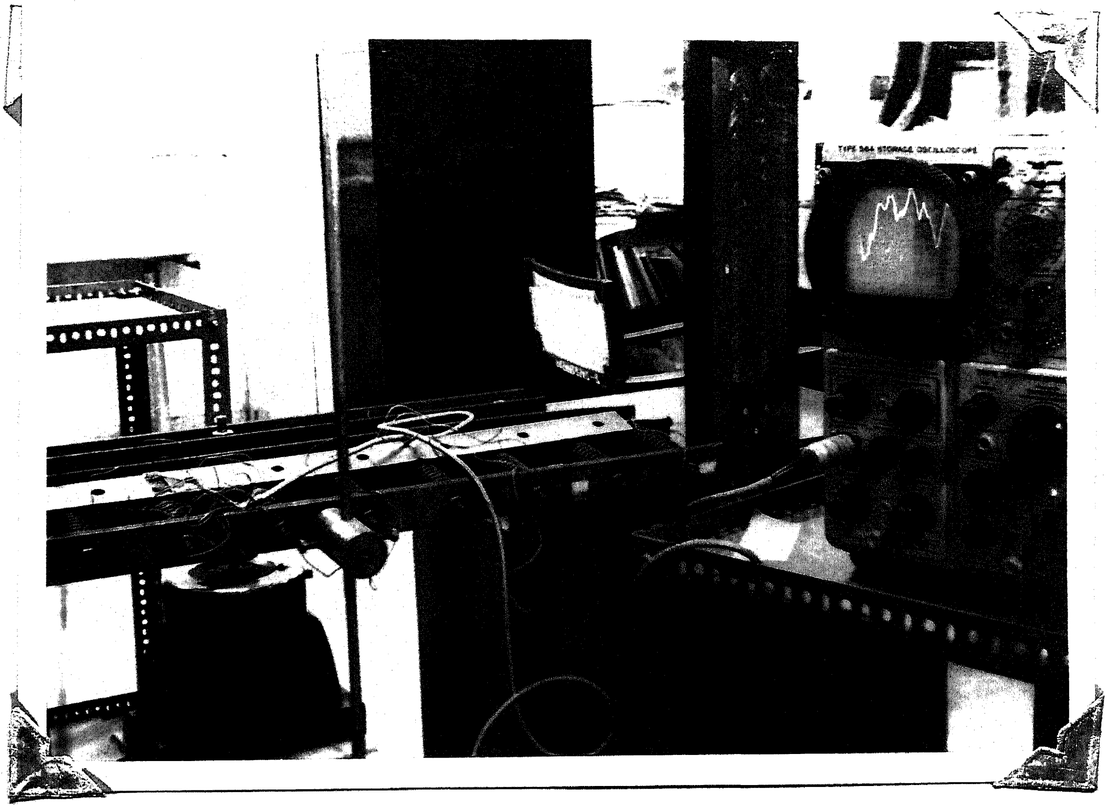


PLATE 9 SET-UP FOR STRAIN MEASUREMENT

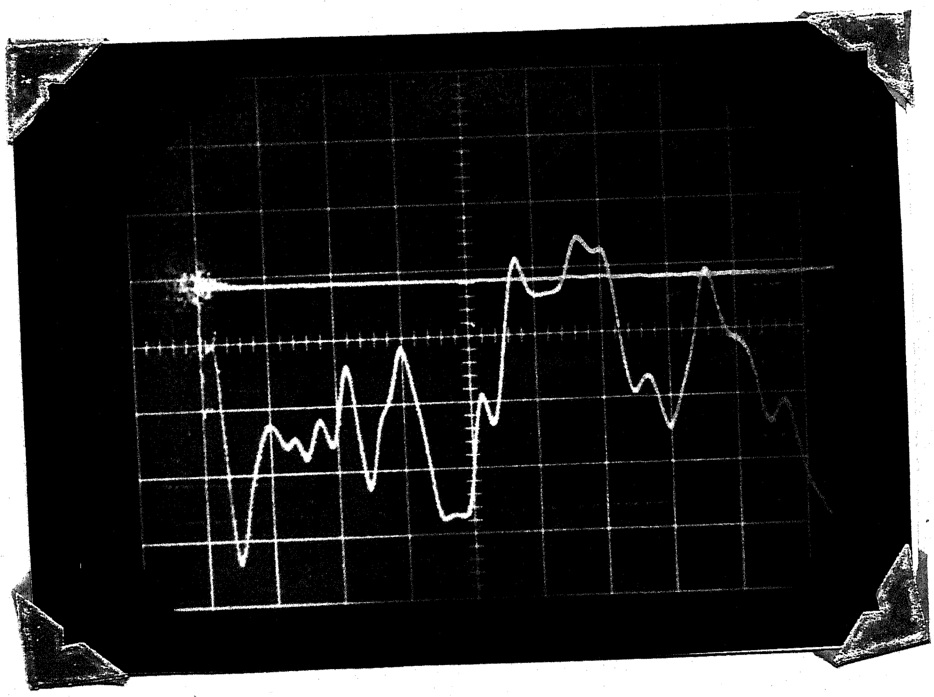
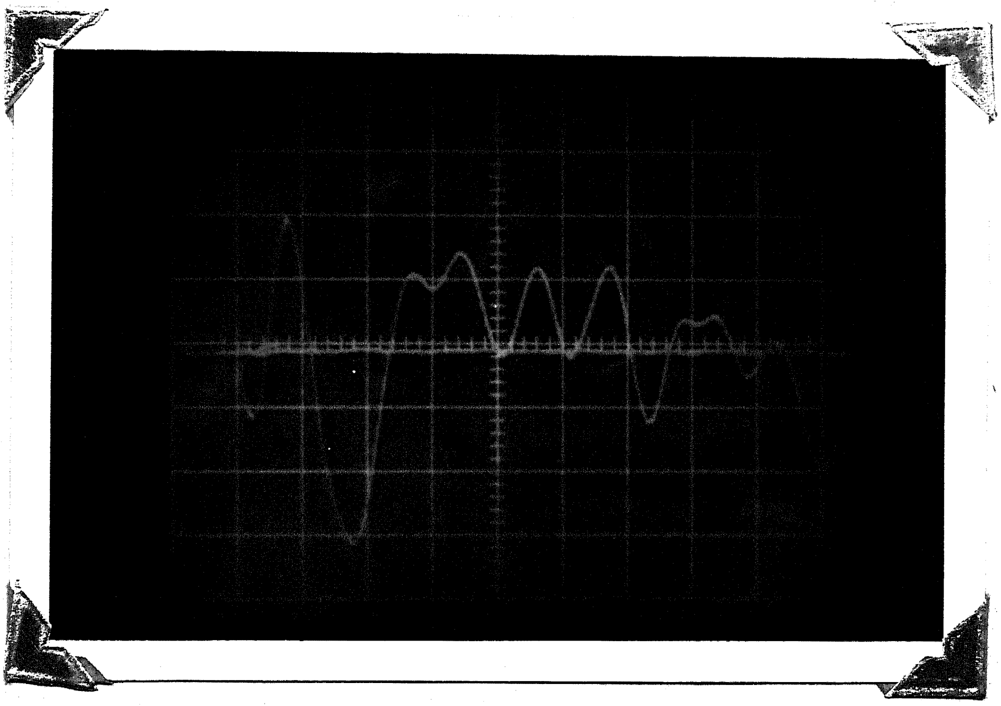


PLATE 10 STRAIN- TIME TRACE

AT $x = L/2$



AT $x = L/5$

PLATE 10 STRAIN TIME TRACE

REFERENCES

1. BRACH, R.H. - "Minimum Design Response for a Class of Simply Supported Beam Shapes. Int. J. Mech. Sciences, 10, 429, 1968.
2. CHERN J.M., DAFALIES J.F., MARTIN J.B. : "Structural Design for Bounds on Dynamic Response" J. ASCE. Engg Mech. DIVISION, Vol. 99, No. EM2, April 1973.
3. DRUCKER, D.C. "A more Fundamental Approach to Stress-Strain Relations" Proceedings First U.S.National Congress of Applied Mechanics ASME 1952 pp. 487.
4. DRUCKER, D.C.,"A definition of a Stable inelastic Material". Journal of Applied Mechanics Vol. 26 Trans. ASME, Vol. 81 Series E 1959 pp. 101.
5. DRUCKER, D.C., GREENBERG, H.J., PRAGER, W. : "Extended Limit Design Theorems for Continuous Media" Quarterly of Applied Mathematics, Vol. 9, 1952, pp. 381.
6. DUWEZ, P.E., CLARK, D.S. and BOHNEBLUST, H.F. : "The behaviour of Long Beams under Impact Loading" Journal of Applied Mechanics, 17, 1950, 27.
7. ERINGEN, A.C. "Transverse Impact on Beams and Plates". Journal of Applied Mechanics, 20, 1953, 461.
8. GOLDSMITH, W. "Impact" , Edward Arnold (Publishers) Ltd., London.
9. HATEYNI, K., "Beams on Elastic Foundation", Michigan Press 1958.
10. HERTZ, H., 'Über die Berührung fester elastischer Körper' , J. J. reine argew. Math (Crelle) 92, 1881, 155.

11. HILDEBRAND, B. FRANCIS : "Methods of Applied Mathematics" Prentice-Hall of India, Private Ltd., New Delhi 1972.

12. HORST SCHWEIGER, : "A Simple Calculation of the Transverse Impact on Beams and its Experimental Verification", Exp. Mechanics, Nov. 1965, p. 373.

13. HORST SCHWEIGER, : "Central Deflection of a Transversely Struck Beam", Exp. Mechanics, April 1970, p. 166.

14. HOPPMANN, W.H., Jr., : "Impact of a Mass on a Damped Elastically Supported Beam", Trans. ASME, 70, 1948, A-125.

15. HOPPMANN, W.H. Jr., : "Impact of a Mass On a Column" , Journal of Applied Mechanics, 16, 1949, 370.

16. ICERMAN, L.J., : "Optimal Structural Design for Given Dynamic Deflection", Int. J.Solids & Structures, 5, 473, 1969.

17. KERR, A.D., :"Elastic and Viscoelastic Foundation Models" Journal of Applied Mechanics, ASME, September 1964 pp 491-498.

18. LEE, E.H. : "The Impact of Mass Striking a Beam", Trans. ASME 1949, Vol. 62, A-129.

19. MARTIN, J.B. : "A displacement bound principle for Plastic Continua Subjected to a Certain Class of Dynamic Loading", Journal of Mech. & Phy. ofSolids Vol. 2, 1964 pp. 165-175.

20. MARTIN, J.B., : "A Displacement bound Principle for inelastic Continua Subjected to Certain Class of Dynamic Loading", Journal of Applied Mechanics, Vol. 4, March 1965.

21. MARTIN, J.B.: "Displacement Bounds for Dynamically Loaded Elastic Structure", Journal of Mech. Engg. Sciences Vol. 10, No. 3, 1968 pp. 213 - 218.

22. MARTIN, J.B. : "On the Application of the Bounding Theorem of Plasticity to Impulsively Loaded Structure", G. Hermen and N. Perrene Eds. Pragman Press N.Y. 1972 pp. 73 - 93.

23. MEIROVITCH, LEONARD, : "Analytical Methods in Vibration", Macmillan Company London, 1967.

24. MIKLOWITZ, J., : "Applied Mechanics Review", 13, 1960, pp. 865 - 878.

25. NOWACKI, W., "Dynamic of Elastic Systems", Chappman & Hall Ltd., London (1963).

26. PLAUT, R.H. : "On minimizing the Response of Structure to Dynamic Loading", Journal of App. Math. & Phy. (ZAMP) Vol. 21, 1970 pp. 1004 - 1010.

27. PLAUT, R.H., FREUND, L.B., : "An Energy Displacement Inequality Applicable to Problems in the Dynamic Stability of Structures", Journal of Applied Mechanics, June 1971, p.536.

28. RAMAN, C.V., :"On Some Application of Hertz Theory of Impact", The Physical Review, Vol. 15, April 1920 pp. 277.

29. ROBINSON, D.N., :"A displacement bound principle for elastic plastic Structure subjected to Blast Loading" J. Mech& Phy.Solids 1970, Vol. 18, pp. 65 - 80.

30. SEILER, J.A., COTTER, B.A. and SYMONDS, P.S. : "Impulsive Loading of Elastic - Plastic Beams", Journal of Applied Mechanics, Vol. 23 Trans. ASME Vol. 78, pp. 515.

31. SYMONDS, P.S., :"Dynamic Load Characteristics in Plastic Bending of Beams", Journal of App. Mech. 20, 1953, 475.

32. SYMONDS, P.S., : "Large Plastic Deformations of Beams under Blast Type Loading", Proc. 2nd, U.S. National Congress of Applied Mechanics, 1954, p. 505.

33. SYMONDS, P.S. and LETH, C.F.A., : "Impact of finite beams of Ductile Metal", Journal of Mech. & Phy. Solids, 2, 1954, p. 92.

34. STEPHENS, R.W.P. & BATE, A.E., : "Wave motion and Sound", Arnold (Publication) London 1950 p. 73.

35. SUNDERARAJAN, V., SRINATH, L.R., JADON, V.K. : "Analysis of Transverse Impact on Beams", Research Report NBS-ME-1 Indian Institute of Technology, Kanpur, August 1971.

36. TIMOSHENKO, S., : "Vibration Problems in Engineering", D.Van Nostrand Co. Inc.N.Y. 1937 p. 349.

37. WINKLER, E. : "Die Lehre Vonder Elastizital Festigkeit", Prague, Czechoslavakia 1867, p. 182.

38. W. PRAGER and R.T. SHIELD, : "Optimal Structural Design for Given Deflection", J. App. Math. and Phy. (ZAMP) 21, 513-523, 1970.

39. ZEENER, C. & FREHBACH, H., : A method of Calculating Energy Losses during Impact, " Trans. ASME 1939, A-67.

BIBLIOGRAPHY

1. HALL, R.G., AL-HASSANI, S.T.S., HALL, R.G., JOHNSON, W., "The Impulsive Loading of Cantilivers" Int. J. Mechanical Sciences, Pergamon Press, 1971, Vol. 13, pp. 415-430.

2. AL-HASSANI, S.T.S. and JOHNSON, "The magnetomotive Loading of Cantilevers, Beams and Frames", Int. J. Mechanical Sciences, Pergamon Press, 1970, Vol. 12, pp. 711 - 722.

3. AL-HASSANI, HASHMI, S.J., JOWNSON, W., "Large Deflection Elastic - Plastic Response of Certain Structures to impulsive load," Int. J. Mechanical Sciences, 1972, Vol. 4, pp. 843-860.

4. BERNAHART AND GOLDSMITH, W., "Stresses in Beams during Transverse Impact", Journal of Applied Mechanics, 24, 1957, p. 440.

5. PARLEY, I.R. and GUSTIN, M.E., "The Vibration Response of a Linear Undamped System Resting on a Non-Linear Springs", Journal of Applied Mechanics, ASME, Vol. 27, 1960, pp. 272-274.

6. PORTER, B. and BILLET, R.A., "Harmonic and Sub-harmonic vibration of a Continuous System having Non-linear Constants", International Journal of Mechanical Sciences, Vol. 7, p. 1965.

7. SRINIVASAN, A.V.,"Steady State Response of Beams Supported on Non-linear Springs", AIAA, J. Vol. 4, No. 10, Oct. 1966 pp. 1863-1869.

8. TACHERT, T.R. and ARYE, R.S., "Shock Response of a Simple Beam on Non-linear Supports", Journal of EM Division ASCE Vol. 91, No. EM6 Dec. 1965 pp. 91-109.

9. HOLZER, S.M., "Response Bounds of Columns with Transient Load", JAM.Vol.38, Series E, No.1, Trans. ASME, March 1971. p.157-161.

10. KROPS, R.J. and WILES, E.W., "On Morchen's Theorem for Stability of Continuous System", International Journal of Engg. Sciences Vol. 4, 1966, pp. 303-329.

11. DICKSSON, J.R. and CAGHEY, T.K., "Stability of Continuous Dynamic System With ParametricExcitation", JAM Vol. 36, June 1967 pp. 212-216.

12. HETEYNI, M., "Series Solution for Beams on Elastic Foundation", JAM, June 1971, pp. 501.

13. BONDER, S.R. and SYMONDS, P.S., "Experimental and Theoretical Investigation of the Plastic Deformation of Cantilever Beam Subjected to Impulsive Loading" Journal of App. Mech. Vol. 29, Dec. 1962, pp. 719.

14. HODGE, P.G., "Plastic Analysis of Structures", Mc-Graw Hill Book Company,Inc. 1959.

15. VLASOV, V.Z. and LEONTOV, U.N., "Beam, Plates and Shells on Elastic Foundation", (Translated from Russian) NASA, TTF-357, 1966.

16. DAILY, J.W. and RILEY, W.F., "Experimental Stress Analysis", McGraw Hill Book Company, Tokyo.

17. KAMESHWAR RAO, N.S.V., DAS, Y.C., AANANDKRISHNAN, M., "Variational Approach : To Beams on Elastic Foundations" Journal of Engg. Mech. Division, ASCE, April 1971, 8034.

18. PRASAD, BIRENDRA and SUNDERAJAN, V., "Studies of the Blast and Impact Loading on Beams", Research Report NBS-ME-3 I.I.T. Kanpur, 1971.

19. THOMSON, T. WILLIAM, "Vibration Theory and Applications", GEORGE ALLEN & UNWIN LTD., LONDON.

20. ANDERSON, A., ROGER, "Fundamentals of Vibrations", Amerind Publishing Co. Ltd., New York.

21. CRANDALL, H., STEPHAN, "Engineering Analysis ", McGraw-Hill Book Company, New York, 1956.

22. ALVERSION, R.C., "Impact with Finite Acceleration Twice of Elastic-Plastic Beams", Journal of Applied Mechanics, 411.